# EECS 127

**Admin ①:** April 30, during class time

**Admin ②!** Project will have extra credit that can be used to clobber midterm + final

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## Material

1. 1D version of LASSO
2. Co-ordinate descent algorimth
3. Demo
4. Quadiatic Programs

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## LASSO Problem.

$$\min_{\vec{x}} \|A\vec{x} - \vec{b}\|_2^2 + \underbrace{\lambda \cdot \|\vec{x}\|_1}_{\text{regularizer}}$$

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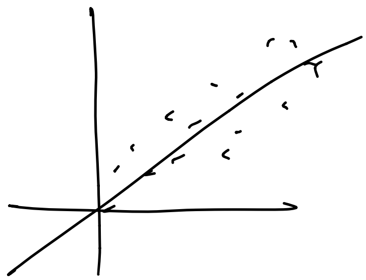
## Simplest 1D regression case.

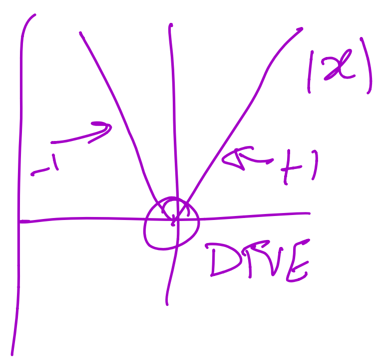
$$\min_{x} \frac{1}{2}\sum_{i=1}^{n} (a_i x - b_i)^2 + \lambda |x|$$

$x$: scalar; $a_i$: scalar; the whole expression: $f(x)$.

$(a_i, b_i) \longrightarrow$ find $x$ that best fits.



CONVEX




$$\frac{d}{dx}|x| = 1 \text{ if } x>0$$
$$= -1 \text{ if } x<0$$
$$\text{DNE } x=0$$

$$\frac{d}{dx} f(x)$$

$$f(x) = \begin{cases} \frac{1}{2}\sum_{i=1}^{n} (a_i x - b_i)^2 + \lambda x & \text{if } x>0 \\ \text{——— " ———} - \lambda x & \text{if } x<0 \\ \text{——— " ———} & \text{if } x=0 \end{cases}$$

usual calculus, but we made cases

$$\frac{d}{dx} f(x) = \begin{cases} \cancel{\frac{1}{2}} \cdot \sum_{i=1}^{n} \cancel{2}(a_i x - b_i)\, a_i + \lambda & \text{if } x>0 \\ \sum_{i=1}^{n} (a_i x - b_i)\, a_i - \lambda & \text{if } x<0 \end{cases}$$

$$\frac{1}{2}\sum_{i=1} 2(a_i x - b_i)a_i - \lambda = 0$$

$\boxed{\text{DNE}}$

$x = 0$

we are trying to find $x$

Case 1: Consider $x > 0$.

$x$ is not known yet!

$$\left(\sum_{i=1}^{n} a_i^2\right) \cdot x - \sum_{i=1}^{n} a_i b_i + \lambda = 0$$

to find a critical point.

$$x = \frac{\sum a_i b_i - \lambda}{\sum_{i=1}^{n} a_i^2}$$

denom always $> 0$

This calculation was done under the assumption that $x > 0$. We better verify this assumption

We need.

$\Rightarrow \sum a_i b_i - \lambda > 0 \Rightarrow \boxed{\sum a_i b_i > \lambda}$

If $\sum a_i b_i > \lambda$ then

$$\boxed{x = \frac{\sum a_i b_i - \lambda}{\sum a_i^2}}$$

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② Consider $x < 0$ $\boxed{\text{ASSUMPTION}}$

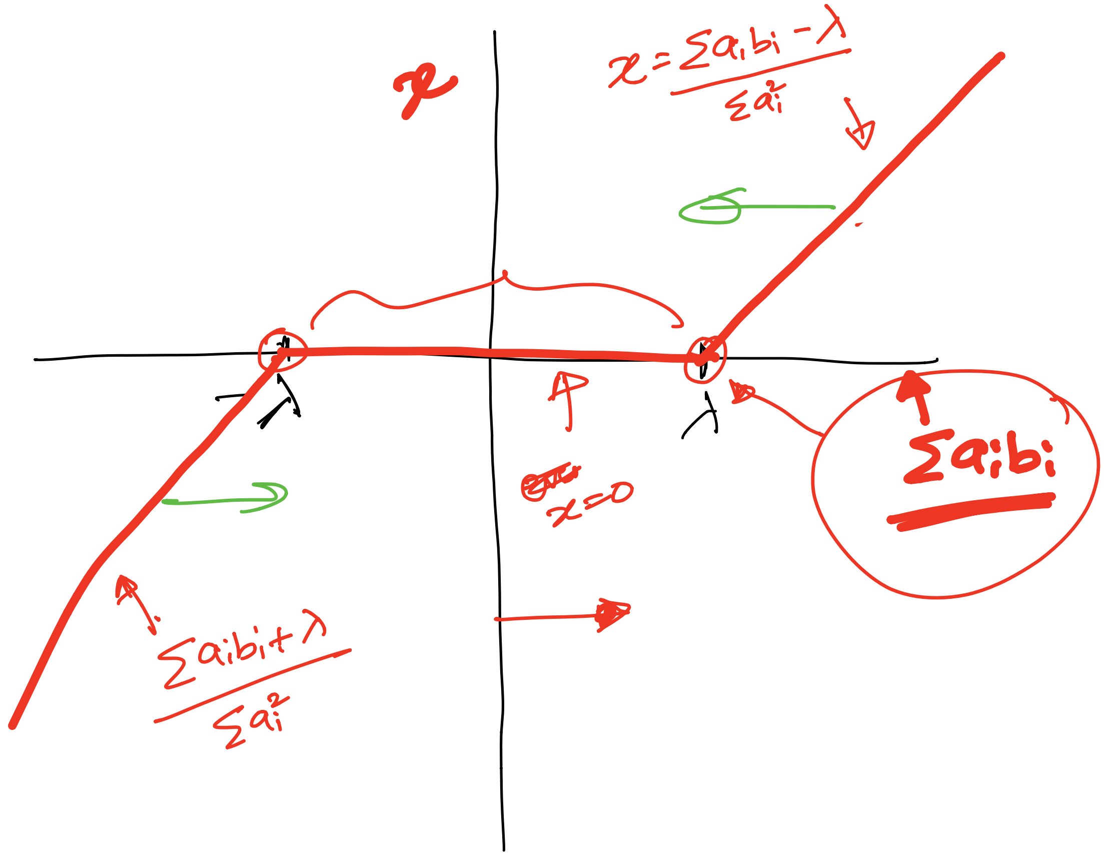
Set $\frac{d}{dx}(\quad) = 0$

$$\Rightarrow \quad x = \frac{\sum_{i=1}^{n} a_i b_i + \lambda}{\sum_{i=1}^{n} a_i^2}$$

Valid only when $\sum_{i=1}^{n} a_i b_i + \lambda < 0$

$$\boxed{\sum a_i b_i < -\lambda.}$$

At all other times, $x = 0$.




"Soft thresholding.

Recall: Standard LS solution:

$$x = \frac{\sum a_i b_i}{\sum a_i^2}$$

$$(A^T A)^{-1} A^T b$$

# Coordinate descent

$\min f(x)$ - function.

$x(0) = [x_1(0), \underbrace{x_2(0) \ldots, x_n(0)}_{\text{fixed}}]$. "guess for the minimum"

$k=1$

$x_1(k) = \underset{x_1}{\text{argmin}}\ f([\underset{\text{var}}{x_1}, \overset{\circ}{x_2(k-1)}, \overset{\circ}{x_3(k-1)} \ldots, \overset{\circ}{x_n(k-1)}])$

$x_2(k) = \underset{x_2}{\text{argmin}}\ f([\underset{k}{x_1(k)}, \underset{\text{var}}{x_2}, \underset{k-1}{x_3(k-1)}, \ldots x_n(k-1)])$

$\vdots$

$x_n(k) = \underset{x_n}{\text{argmin}}\ f([x_1(k), x_2(k), \ldots x_{n-1}(k), x_n])$

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$\|Ax - b\|_2^2 + \lambda \|x\|_1$ 2D version

$$\left\| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\|_2^2 + \lambda (|x_1| + |x_2|)$$

$($

$$= (\underline{a_{11}x_1} + \underbrace{a_{12}x_2 - b_1}_{\text{constant}})^2 + (\underline{a_{21}x_1} + \underbrace{a_{22}x_2 - b_2}_{\text{const.}})^2 + \lambda|x_1| + \underbrace{\lambda|x_2|}_{\text{const.}}$$

$\to g(x_1, x_2)$

Start at $(x_1(0), x_2(0))$

Freeze $x_2(0)$

$$\min_{x_1} g(x_1, x_2(0))$$

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LASSO is one example of what is called a Quadratic Program.

$$\min \ \frac{1}{2}\vec{x}^T H \vec{x} + \vec{c}^T\vec{x} + d$$

s.t. $A\vec{x} \le b$

$C\vec{x} = e$

QP: efj ① $H = H^T$

② objective function is CONVEX.

③ Constraints are liner.

Non-convex quadratic : hyperbola.

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Unconstrained quadratic.

$$f(\vec{x}) = \frac{1}{2}\vec{x}^T H \vec{x} + \vec{c}^T \vec{x} + d.$$

$H \succeq 0$
PSD.
Symmetric

Set $\nabla f = 0$

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Equality constrained quadratic.

$$\min \ \frac{1}{2}\vec{x}^T H \vec{x} + \vec{c}^T \vec{x} + d.$$

s.t. $A\vec{x} = b$.

Can be transformed into an

unconstrained quadratic optimization

Constraints:

Feasible: $\{\vec{x} \mid \vec{x} = \vec{x_0} + N\vec{z}\}$

$A\vec{x} = \vec{b}$

$\vec{x_0}$ any solution to $A\vec{x} = \vec{b}$

nullspace of $A$.

$$\min_{\vec{z}} \frac{1}{2}(\vec{x_0} + N\vec{z})^T H (\vec{x_0} + N\vec{z}) + \vec{c}^T(\vec{x_0} + N\vec{z}) + d$$

Quadratic in $\vec{z}$

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