EECS 127

① April 30 5-7:30 : Please let us know of conflicts. by Friday.

② Project: Register Project + team by Friday 24th April.

- LPs
- QPs
- SOCPs

} amenable to solvers.

CVX

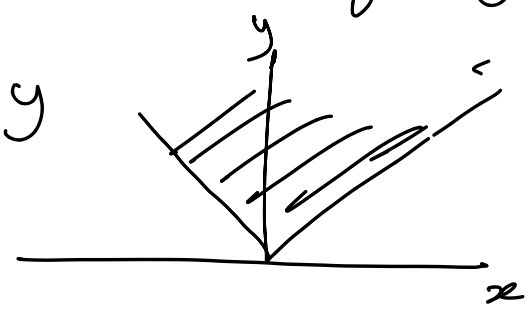
Second Order Cone Programs

Cone: Set of points. $\mathcal{C} \subseteq \mathbb{R}^n$. $\alpha \in \mathbb{R}$

$\vec{x} \in \mathcal{C}$ then $\alpha \vec{x} \in \mathcal{C}$ for $\alpha > 0$.

Convex cone: $\vec{x} + \vec{y} \in \mathcal{C}$ if $\vec{x}, \vec{y} \in \mathcal{C}$

e.g. $|x| \leq y$



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e.g. Polyhedral cone:

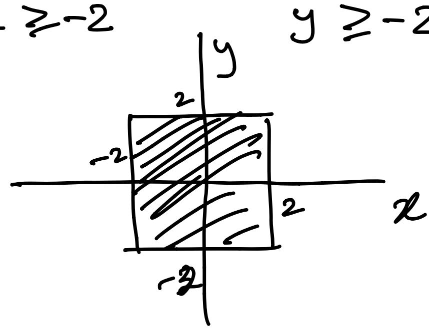
Start : $A\vec{x} \leq \vec{b}$ $\vec{x} \in \mathbb{R}^n$

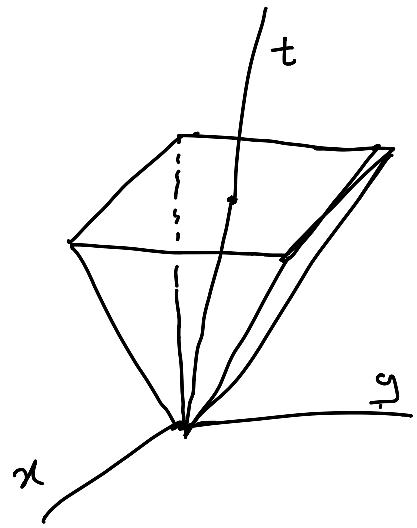
Consider: $\{ A\vec{x} \le \vec{b}t \ , \ t \ge 0 \}$.

$(\vec{x}, t) \longrightarrow$ forms a cone

e.g. $x \le 2$ $\quad y \le 2$

$x \ge -2$ $\quad y \ge -2$





If

$t=1$:

then: I get a "slice" of the polyhedron.

<u>Eg. Ellipsoidal cone.</u>

<u>Ellipsoid:</u> $x^T P x + q^T x + r \le 0$. $\quad P \succ 0$, $x \in \mathbb{R}^n$

$$\| A\vec{x} + \vec{b} \|_2^2 \le c^2$$

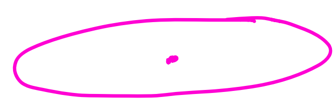
A was full rank.

PD

$$\underbrace{\vec{x}^T A^T A \vec{x} + 2\vec{b}^T A\vec{x} + \vec{b}^T b - c^2 \le 0}$$

$A^T A \succ 0$

Ellipse.



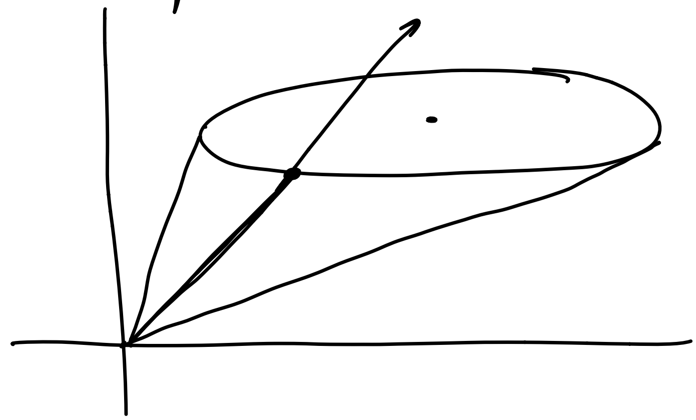
Ellipsoidal cone:

scalar

$$\| A\vec{x} + \vec{b}t \|_2 \le c \cdot t$$

scalar variable.

All $(\vec{x}, t)$ that satisfy the above inequality belong to the ellipsoidal cone.



If $(\vec{x}, t)$ satisfies $\|A\vec{x} + \vec{b}t\|_2 \le ct$.

then $(\alpha \cdot \vec{x}, \alpha t) \Rightarrow$

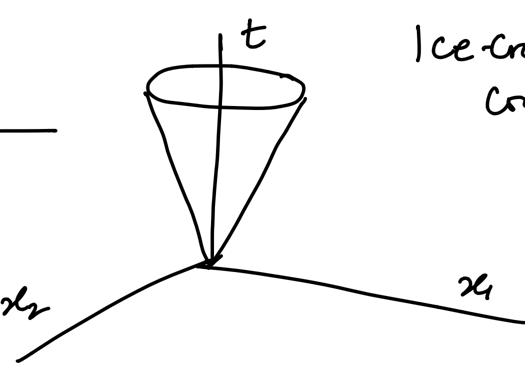
$$\|A\alpha\vec{x} + b\alpha t\|_2 = \alpha \|A\vec{x} + bt\|_2$$

$$\le \alpha \cdot c \cdot t.$$

Second-order cone:

SOC $\in \mathbb{R}^3$ $\quad (x_1, x_2, t) \in \mathbb{R}^3$ s.t.

$$\sqrt{x_1^2 + x_2^2} \le t$$



Ice-cream cone

A SOC in $\mathbb{R}^{n+1}$ is defined as:

$$K_n = \{ (\vec{x}, t) \mid \vec{x} \in \mathbb{R}^n, t \in \mathbb{R}, \|\vec{x}\|_2 \le t \}.$$

2D: (x, t), $x, t \in \mathbb{R}$, $|x| \leq t$

Consider: $\vec{x} \in \mathbb{R}^n$

$(\vec{x}, t)$ that satisfy

Ineq ☆ $\| A\vec{x} + \vec{b}t \|_2 \leq \vec{c}^T x + dt$

vector vector scalar

Is this a cone? in $\mathbb{R}^{n+1}$

α ✱

$(\vec{x}, t) \in C$. $\alpha > 0$.

Consider $(\alpha\vec{x}, \alpha t)$

$\| A\alpha\vec{x} + b\alpha t \|_2 \leq \vec{C}^T \alpha x + d\alpha t$. ✓

If we have: $(\vec{x}_1, t_1)$, $(\vec{x}_2, t_2)$.

$\| A(\vec{x}_1 + \vec{x}_2) + \vec{b}(t_1 + t_2) \|_2$

$\leq \| A\vec{x}_1 + \vec{b}t_1 \|_2 + \| A\vec{x}_2 + bt_2 \|_2$

Ineq ☆ $\leq \vec{c}^T x_1 + dt_1 + \vec{c}^T x_2 + dt_2$.

SOCP:

linear objective.

$$\min \quad \vec{c}^T \vec{x}$$

$$\text{s.t.} \quad \underbrace{\|A_i \vec{x} + \vec{b}\|_2 \le \vec{c}_i^{\,T} \vec{x} + d}_{\text{Constraint}} \qquad \forall i = 1, 2, \ldots m.$$

" Second-order cone constraint".

$t = 1$ (Slice)

eg: An LP is an SOCP.

$$\min \quad \vec{c}^T \vec{x}$$

$$\text{s.t.} \quad \vec{a}_i^{\,T} \vec{x} \le b_i \qquad i = 1, \ldots m.$$

$$\min \quad \vec{c}^T \vec{x}$$

$$\|0 \vec{x} + 0\|_2 \le b_i - \vec{a}_i^{\,T} x \qquad i = 1 \ldots m$$

QPs are also SOCPs.

$$\min \quad \vec{x}^T Q \vec{x} + \vec{c}^T \vec{x}. \qquad Q \succeq 0$$

$$\text{s.t.} \quad \vec{a}_i^{\,T} \vec{x} \le b_i \qquad i = 1, \ldots m.$$

Let: $\vec{x}^T Q \vec{x} = y.$

$$\vec{x}^T \underline{Q^{1/2} Q^{1/2}} \vec{x} = y$$

$$\min \quad y + \vec{c}^T \vec{x}$$

$$\text{s.t.} \quad \vec{a_i}^T \vec{x} \leq b_i \qquad i = 1 \dots m$$

$$\text{and} \quad \underline{\vec{x}^T Q \vec{x} = y}$$

↳ By observation / magic trick.

$$\left\| \begin{bmatrix} 2 Q^{1/2} x \\ y-1 \end{bmatrix} \right\|_2 \leq y+1$$

} algebra will show that this is equivalent to $\vec{x}^T Q \vec{x} \leq y$;

$$\min \quad y + \vec{c}^T \vec{x}$$

$$\text{s.t.} \quad \vec{a_i}^T x \leq b_i \qquad i = 1 \dots m$$

$$\text{and} \quad \underbrace{\left\| \begin{bmatrix} 2 Q^{1/2} x \\ y-1 \end{bmatrix} \right\|_2 \leq y+1}$$

<u>Example:</u>

$$\min_{\vec{x}} \sum_{i=1}^{m} \| A_i \vec{x} - \vec{b}_i \|_2$$

Is this a quadratic program?

$$\min_{\vec{x}} \|A\vec{x}-\vec{b}\|_2 \iff \min_{\vec{x}} \|A\vec{x}-\vec{b}\|_2^2$$

QP

We can think of this as an SOCP.!

$$y_i = \|A_i \vec{x} - \vec{b}_i\|_2$$

linear

$$\min_{\vec{y}} \sum_{i=1}^{n} y_i$$

$$\text{s.t. } \|A_i\vec{x} - b_i\|_2 \le y_i \qquad i=1 \dots n.$$

Example: $\min_{\vec{x}} \left[\max_{\underline{i=1,\dots p}} \|A_i\vec{x} - \vec{b}_i\|_2 \right]$

$\Updownarrow$

$$\min_{y} \quad y.$$

$$\text{s.t. } \|A_i\vec{x} - \vec{b}_i\|_2 \le y \qquad i = 1, 2, \dots p.$$

SOCP

Facility location Problems

Where to put emergency room, playground etc?

$$\min_{x} \quad \max_{i=1\dots p} \quad \|\vec{x} - \vec{y}_i\|_2^2$$

locations of population centers.

min average distance travelled.

$$\min_{x} \frac{1}{m} \sum_{i=1}^{m} \|\vec{x} - \vec{y}_i\|_2$$

Newton's Method

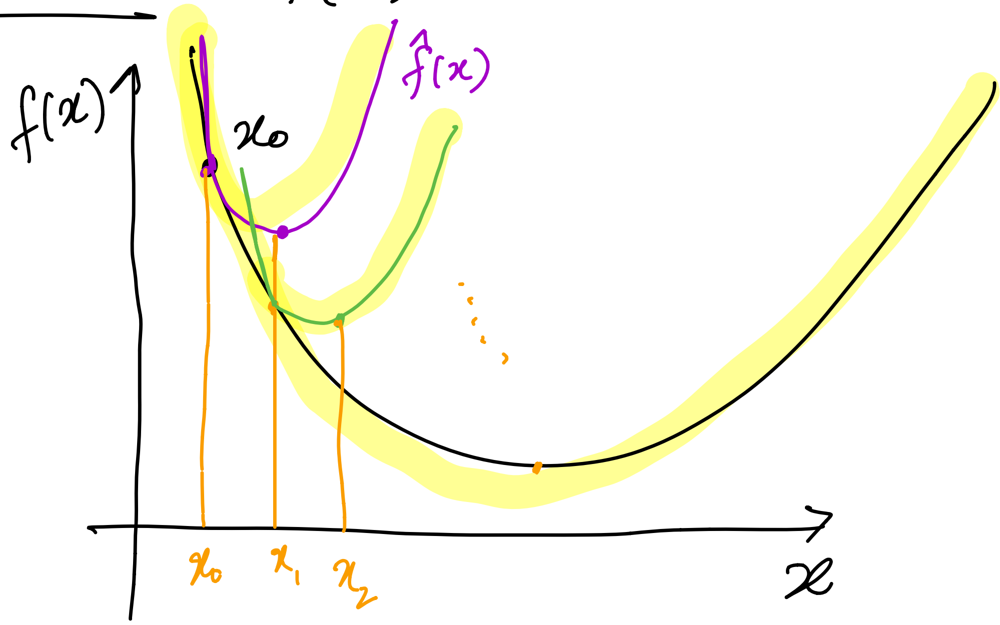
→ Iterative method, like gradient descent.

→ Key Idea: Approximate you function locally as a quadratic.

Find the minimum of the quadratic

Iterate

Consider: $f(x)$. $x \in \mathbb{R}$



Want: $\vec{x}_0, \vec{x}_1, \ldots$ that converges to the opt $\vec{x}^*$

How do we do this?

$$f(\vec{x}+\vec{v}) = f(\vec{x}) + \nabla f(\vec{x})^T \vec{v} + \frac{1}{2} \vec{v}^T \boxed{\nabla^2 f(\vec{x})} \vec{v} + \ldots$$

H.O.T.

Taylor's thm.

Quadratic approximation

How to find min of quadratic?

Hessian Positive definite. $H \succ 0$

Any general quadratic:

$$g(\vec{v}) = \frac{1}{2}\vec{v}^T H \vec{v} + \vec{c}^T v + d.$$

$$\nabla g(\vec{v}) = 0$$ to find min

$$\vec{v} = -H^{-1}\vec{c}$$ minimizer.

So the $\vec{v}$ that minimizes our quadratic is:

$$\underbrace{\vec{v} = -\left[\nabla^2 f(\vec{x})\right]^{-1}\nabla f(\vec{x})}_{\text{Newton step direction.}}$$

$$H(\vec{x}) = \nabla^2 f(\vec{x})$$

$$\underbrace{\vec{x}_{k+1} = \vec{x}_k - \left[H(x_k)\right]^{-1}\nabla f(\vec{x}_k)}_{\text{Newton iteration.}}$$

Damped Newton's method:

$$\vec{x}_{k+1} = \vec{x}_k - \underbrace{\eta}_{\text{stepsize.}} \cdot \left[H(x_k)\right]^{-1} \nabla f(\vec{x}_k)$$

- (1) If f is quadratic

$\implies$ N M will get to minimum in one step!

eg. $f(\vec{x}) = \|\vec{x}\|. = x_1^2 + x_2^2.$

$$\nabla f(\vec{x}) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\nabla^2 f(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

inv. hessian ↙ ↙ grad

$$\vec{u} = - \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}.$$

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

↑ minimizer!

<u>Advantange:</u>

(2) Often be faster than GD.

(3) <u>Disadvantage:</u> NM: H^{-1} computation can be challenging / unstable.