EECS 127 Lecture 7.

$A \in \mathbb{R}^{m \times n}$. $m > n$.

$$A_k = \sum_{i=1}^{k} \sigma_i \vec{u_i} \vec{v_i}^T \quad ; \quad A_k = \underset{\substack{B \in \mathbb{R}^{m \times n} \\ \text{Rank}(B) = k.}}{\text{argmin}} \|A - B\|_F^2$$

$$\sigma_1 \geq \sigma_2 \cdots \geq \sigma_n > 0.$$

Proof: Consider: $\|A - A_k\|_F^2 = \left\| \sum_{i=1}^{n} \sigma_i \vec{u_i} \vec{v_i}^T - \sum_{i=1}^{k} \sigma_i \vec{u_i} \vec{v_i}^T \right\|_F^2$

$$= \left\| \sum_{i=k+1}^{n} \sigma_i \vec{u_i} \vec{v_i}^T \right\|_F^2 = \sum_{i=k+1}^{n} \sigma_i^2$$

Recall: Frob. Norm is invariant to ortho~~gonal~~normal transformations. $\|A\|_F = \|AU\|_F$

$$\|A\|_F^2 = \|U \Sigma V^T\|_F^2 = \|\Sigma\|_F^2 = \sum_{i=1}^{n} \sigma_i^2$$

So we want to show:

For any B, that is rank(k) ; $\|A - B\|_F^2 \geq \|A - A_k\|_F^2 = \sum_{i=k+1}^{n} \sigma_i^2$

Notation: $\sigma_i(A)$: ith singular value of A

$\sigma_i(A-B)$: –"— " —— $A-B$

$$\|A-B\|_F^2 = \sum_{i=1}^{n} \underbrace{\sigma_i^2(A-B)}$$

all sing. values of $A-B$.

$\boxed{\geq}$ WANT

$$\|A-A_k\|_F^2 = \sum_{i=k+1}^{n} \sigma_i^2(A)$$

$(n-k)$ smallest SV of A.

So if we could show:

$$\sigma_i(A-B) \geq \sigma_{k+i}(A)$$

How to show this?

Rank constraint.

Ingredients:

- Spectral norm.
- B has rank k. $\Rightarrow \sigma_{k+1}(B) = 0$.
- Triangle ineq.

$\sigma_{k+i}(A) = k+i$ th largest singular value of A.

$=$ largest SV after removing the top $k+i-1$ singular values.

$= \|A - A_{k+i-1}\|_2$ $\quad$ A_{k+i-1} is the best rank $(k+i-1)$ approximation to A in the spectral norm sense.

$\sigma_i(A-B)$

$A-B=C$

$\sigma_i(C) = \|C - C_{i-1}\|_2$: C_{i-1} : Best rank $i-1$ approx to C

$\sigma_{k+1}(B) = 0.$

$= \|B - B_k\|_2$ B_k : Best rank k approx. to B in spectral norm sense.

$\sigma_{k+i}(A) = \|A - A_{k+i-1}\|_2$

$= \|B+C - A_{k+i-1}\|_2$

$$\sigma_i(A-B) = \sigma_i(C)$$
$$= \|C - C_{i-1}\|_2 + \|B - B_k\|_2$$
$$\geq \|C + B - C_{i-1} - B_k\|_2$$

(Triangle inequality for spectral norm)

$$= \|A - C_{i-1} - B_k\|_2$$
$$= \|A - D\|_2$$
$$\geq \|A - A_{k+i-1}\|_2.$$
$$= \sigma_{k+i}(A).$$

$$A_{k+i-1} = \underset{\substack{\mathrm{Rank}(D) \\ \leq k+i-1}}{\mathrm{argmin}} \|A - D\|_2$$

Let $\underbrace{C_{i-1}}_{\text{rank } i-1} + \underbrace{B_k}_{\text{rank } k} = D$

$\mathrm{Rank}(D) \leq \mathrm{Rank}(C_{i-1}) + \mathrm{Rank}(B_k)$

$\leq k+i-1$