EECS 127

Lecture 14, March 5 2020

Today: More duality.

Minimax Thm.

Slaters Conditions

LP duals.

"Shadow" prices

$$p^* = \min f_0(\vec{x})$$
$$f_i(\vec{x}) \le 0 \qquad 1 \le i \le m$$
$$h_i(\vec{x}) = 0. \qquad 1 \le i \le p$$

Primal problem.

$a^T x$

$= [a_1 \ a_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= b = a_1 x_1 + a_2 x_2$

$$d^* = \max_{\substack{\vec{\lambda} \ge 0 \\ \vec{\nu}}} g(\vec{\lambda}, \vec{\nu})$$

Dual problem.

$$g(\vec{\lambda}, \vec{\nu}) = \min_{\vec{x}} L(\vec{x}, \vec{\lambda}, \vec{\nu})$$
$$= \min_{\vec{x}} f_0(\vec{x}) + \sum_{i=1}^{m} \lambda_i f_i(\vec{x}) + \sum_{i=1}^{p} \nu_i h_i(\vec{x})$$

Weak duality: $d^* \le p^*$ $\qquad$ $p^* - d^*$: duality gap.

Thm: Minmax theorem.

For any sets $\mathcal{X}, \mathcal{Y}$ and any function F.

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} F(x,y) \geq \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} F(x,y)$$

Proof: Fix $x_0 \in \mathcal{X}$, $y_0 \in \mathcal{Y}$.

Define:
$$h(y_0) := \min_{x \in \mathcal{X}} F(x, y_0)$$
$$g(x_0) := \max_{y \in \mathcal{Y}} F(x_0, y)$$

$$h(y_0) = \min_{x \in \mathcal{X}} F(x, y_0) \leq \underbrace{F(x_0, y_0)}_{\text{specific realization}} \leq \max_{y \in \mathcal{Y}} F(x_0, y) = g(x_0)$$

$$\Rightarrow \forall x_0, y_0 \Rightarrow h(y_0) \leq g(x_0)$$

$$\max_{y_0 \in \mathcal{Y}} h(y_0) \leq \min_{x_0 \in \mathcal{X}} g(x_0)$$

$$\max_{y_0 \in \mathcal{Y}} \min_{x \in \mathcal{X}} F(x,y) \leq \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} F(x,y)$$

Connecting back.

$$p^* = \min_{f_i(x) \le 0} f_0(x)$$

Consider: $\max_{\lambda \geq 0} L(x,\lambda) = \max_{\lambda \geq 0} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \right)$

$$= \begin{cases} \infty & \text{if for some } i,\ f_i(x) > 0. \\ f_0(x) & \text{if } f_i(x) \le 0 \quad \forall i \end{cases}$$

$$p^* = \min_{x} \max_{\lambda \geq 0} L(x,\lambda)$$

$$d^* = \max_{\lambda \geq 0} g(\lambda) = \max_{\lambda \geq 0} \min_{x} L(x,\lambda)$$

$$p^* \geq d^*$$

- **Strong duality.** $p^* = d^*$ (Only sometimes).

Slater's condition.

$$\min f_0(x) = p^*$$
$$f_i(x) \leq 0$$
$$h_i(x) = 0$$

$\boxed{\text{CONVEX}}$

AND. $\exists$ a point x_0 such that. $x_0 \in \text{relint}(D)$

$f_i(x) < 0$ for all $1 \leq i \leq m$ (strictly feasible).

then strong duality holds

$\boxed{\text{CONVEX}}$

Refined Slaters condition.

$f_1, f_2 .. f_k$ are affine conditions.

Strong duality holds if $\exists\, x_0$ s.t.

$f_i(x_0) \leq 0$ $\quad i = 1, \dots k$

$f_i(x_0) < 0$ $\quad i = k+1, \dots m.$

Dual of an LP . Linear program .

$$\min \quad \vec{c}^{\,T} \vec{x} \qquad \text{Primal}$$
$$\text{s.t.} \quad A\vec{x} \preceq \vec{b}$$

$$L(\vec{x}, \vec{\lambda}) = \vec{c}^{\,T} \vec{x} + \vec{\lambda}^T (A\vec{x} - \vec{b})$$
$$= (A^T \vec{\lambda} + \vec{c})^T \vec{x} - \vec{\lambda}^T \vec{b}$$

$$g(\vec{\lambda}) = \min_{\vec{x}} \; L(\vec{x}, \vec{\lambda})$$
$$= \begin{cases} -\infty & A^T \lambda + c \neq 0 \\ -\vec{b}^{\,T} \lambda & A^T \lambda + c = 0 \end{cases}$$

<u>Dual</u>: $\max\limits_{\vec{\lambda} \succeq 0} \; g(\vec{\lambda}) = \boxed{\begin{aligned} &\max_{\lambda \succeq 0} \; -\vec{b}^{\,T} \vec{\lambda} \\ &\text{s.t. } A^T \vec{\lambda} + \vec{c} = 0 \end{aligned}}$ dual.

Economic / Pricing interpretation of duality.

Winemaking business:

200 kilos merlot grapes.

300 kilos shiraz grapes.

Redilicious: Blend 1:	4 kilos merlot + 1 kilo shiraz.	q_1	\$20
Bluelicious: Blend 2:	2 kilos merlot + 3 kilos shiraz.	q_2	\$15

Maximize: $20q_1 + 15q_2$

$4q_1 + 2q_2 \leq 200$ Total merlot

$q_1 + 3q_2 \leq 300$ total shiraz.

What if you wanted to just sell off the grape?

λ_1: for merlot grapes

λ_2: for shiraz grapes.

$$\max_{q_1, q_2} \; 20q_1 + 15q_2 + \lambda_1(200 - 4q_1 - 2q_2) + \lambda_2(300 - q_1 - 3q_2)$$

$$= \max_{q_1 q_2} \; (20 - 4\lambda_1 - \lambda_2) q_1 + (15 - 2\lambda_1 - 3\lambda_2) \cdot q_2 + 200\lambda_1 + 300\lambda_2$$

If $20 - 4\lambda_1 - \lambda_2 < 0$ don't do it.

$15 - 2\lambda_2 - 3\lambda_2 < 0$

If $\left.\begin{array}{l} 20 - 4\lambda_1 - \lambda_2 = 0, \\ 15 - 2\lambda_1 - 3\lambda_2 = 0 \end{array}\right\}$

Profit lowerbound: $\boxed{\begin{array}{l} \min\limits_{\lambda_1, \lambda_2} \; 200\lambda_1 + 300\lambda_2 \\ \text{s.t. } 20 - 4\lambda_1 - \lambda_2 = 0 \\ \quad 15 - 2\lambda_1 - 3\lambda_2 = 0 \end{array}}$ Dual

$\min \ f_0(x)$.

s.t. $f_i(x) \le 0$

Change of variable.

$\min \ t$

$(u,t) \in G$

$u \le 0$

$\min \ x^2$ $\quad f_0(x)$

s.t. $x + 3 \le 0$. $\quad f_1(x)$

$x \quad (x+3, x^2)$

(u, t)

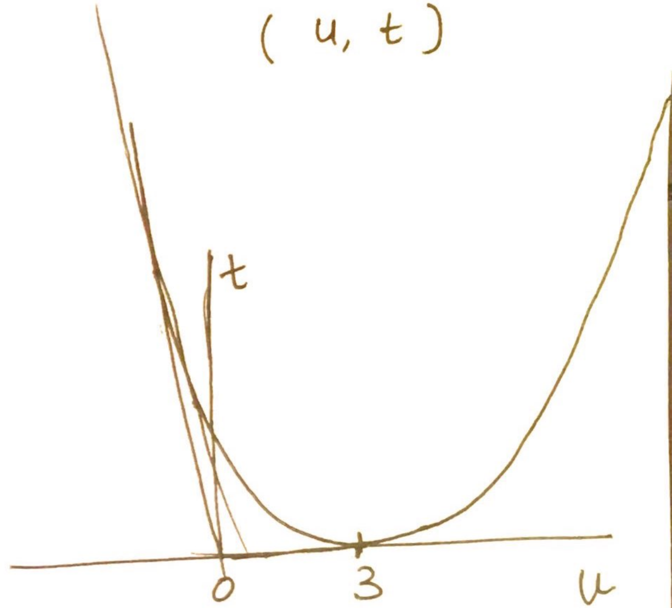
$G: \{ (f_1(x), f_0(x)) \in \mathbb{R}^2 \}$

$p^* = \min \{ t \mid (u,t) \in G, \ u \le 0 \}$.

$L(u, t, \lambda) = t + \lambda u = b$

$t = -\lambda u + b$

slope $\quad$ intercept



Line: $y = mx + c$

$u = x + 3$

$t = x^2$