EECS 127

March 31, 2020

Today: Linear Programs.

e.g. $\min \quad 2x - 3y.$
s.t. $x + y \leq 2$
$x - y = 4$

General: $\min \quad \vec{c}^T \vec{x}$
s.t. $A\vec{x} \preceq \vec{b}$

- Any equality constraint: $a_i^T x = b_i$ can be written as $a_i^T x \leq b_i$ and $a_i^T x \geq b_i$
  So equality constraints can be looped into inequality constraints.
- $a_i^T \vec{x} \geq b_i$ can be written as $-a_i^T \vec{x} \leq -\vec{b}_i$

Standard Form:

$\min \quad \vec{c}^T \vec{x}$
s.t. $A\vec{x} = \vec{b}.$
$\vec{x} \geq 0.$

All LPs can be translated to standard form.

(1) Eliminate inequality constraints by introducing slack variables.
e.g. Consider a constraint of the form:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

Rewrite: $$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i \quad ; \quad s_i \geq 0.$$

s_i : slack variable.

2. Eliminate free variables.

How do we get $x_i \geq 0$ for every variable?

If you have some x_j that is unconstrained, then write $x_j = x_j^+ - x_j^-$, where $x_j^+ \geq 0$, $x_j^- \geq 0$.

Any real number can be written as the difference of two non-negative numbers.

Example:

$$\begin{aligned} \min.\quad & 2x_1 + 4x_2. \\ \text{s.t.}\quad & x_1 + x_2 \geq 3. \\ & 3x_1 + 2x_2 = 14 \\ & x_1 \geq 0 \end{aligned}$$

$x_2 = x_2^+ - x_2^-$

$x_2^+ \geq 0$, $x_2^- \geq 0$.

$x_1 + x_2 + x_3 = 3$.

$x_3 \geq 0$.

$$\rightarrow \quad \begin{aligned} \min.\quad & 2x_1 + 4x_2. \\ \text{s.t.}\quad & x_1 + x_2^+ - x_2^- + x_3 = 3. \\ & 3x_1 + 2x_2^+ - 2x_2^- = 14 \\ & x_1, x_2^+, x_2^-, x_3 \geq 0. \end{aligned}$$

Standard form is computationally more convenient.

Graphical Representation.

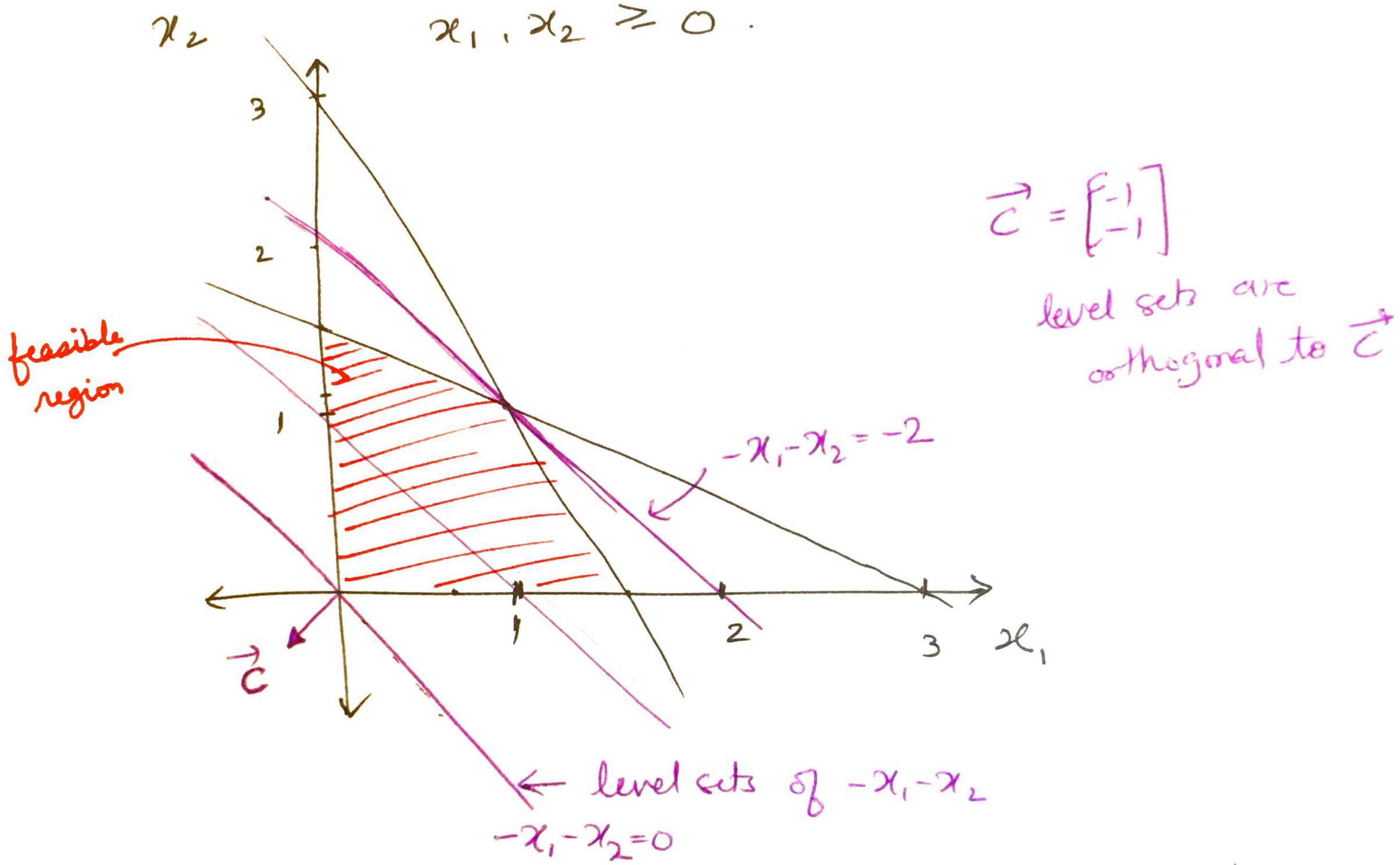
Example:

$$\text{minimize} \quad -x_1 - x_2$$

$$\text{s.t.} \quad x_1 + 2x_2 \leq 3 \quad (1)$$

$$2x_1 + x_2 \leq 3. \quad (2)$$

$$x_1, x_2 \geq 0.$$



We keep moving in the direction of $-\vec{c}$ until we cant.

We hit a "corner".

Def^n: Polyhedron:

Set: $\{x \in \mathbb{R}^n \mid A\vec{x} \geq \vec{b}\}$. $A \in \mathbb{R}^{m \times n}$, $\vec{b} \in \mathbb{R}^m$

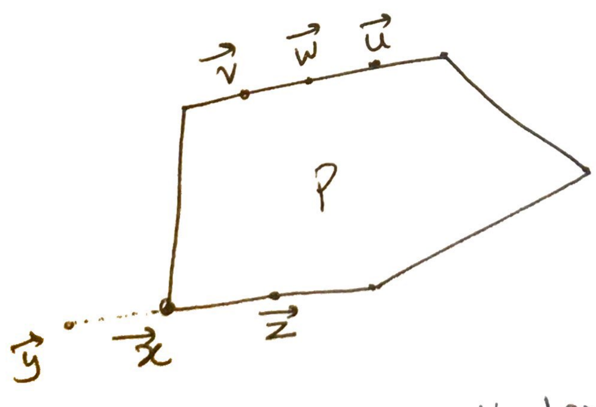
Can also be written in "standard form" with equality constraints:

set: $\{\vec{x} \in \mathbb{R}^\ell \mid C\vec{x} = \vec{d}, \vec{x} \geq 0\}$.

using earlier conversion. # of variables might change.

Def^n: Extreme point. P: Polyhedron. (Also vertex)

$x \in P$ is an extreme point of P if we cannot find two vectors $\vec{y}, \vec{z} \in P$ both different from $\vec{x}$ and a scalar $\lambda \in [0,1]$, such that $\vec{x} = \lambda\vec{y} + (1-\lambda)\vec{z}$



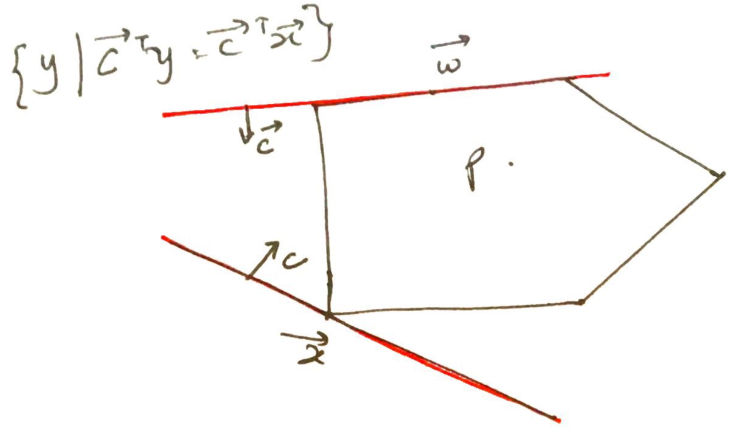
$\vec{x}$ has to be a convex combination of $\vec{y}, \vec{z} \in P$ to not be extreme.

Def^n: Extreme point (Also Vertex).

$\vec{x} \in P$ is a vertex/extreme point of P if there exists some $\vec{c}$ such that $\vec{c}^T\vec{x} < \vec{c}^T\vec{y}$ for all $\vec{y} \in P, \vec{y} \neq \vec{x}$

i.e. All of P should be on "one side" of the hyperplane $\{\vec{y} \mid \vec{c}^T\vec{x} = \vec{c}^T\vec{y}\}$

Hyperplane should meet P at only one point.

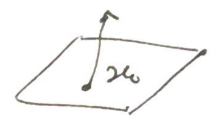


General form of hyperplane

$\vec{c}^T(\vec{x} - \vec{x_0}) = 0$

$\vec{c}$: normal vector

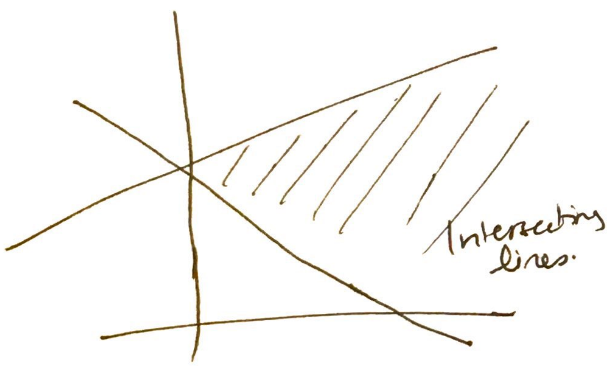
$\vec{x_0}$: point on hyperplane.

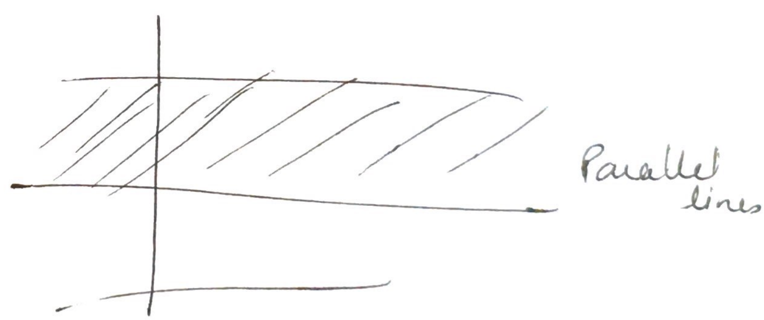


Fact: If P has an extreme point, ~~then~~ $\Longleftrightarrow$ P does not "contain a line"

$P\{x \mid A\vec{x} \le \vec{b}\}$.

Intuition in 2D.





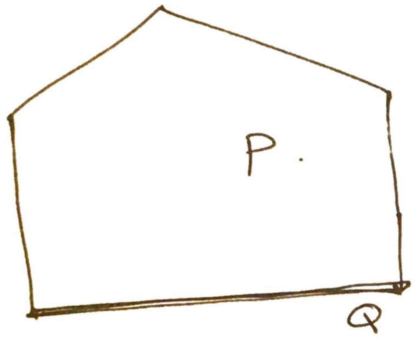
Thm: Consider a linear programming problem of minimicing $c^T x$ over poly. P. $\min c^T x$. s.t $A\vec{x} \leq \vec{b}$

Suppose P has atleast one extreme point.

Suppose optimal solution exists and is finite

Then, there exists an optimal solution that is an extreme point of P.

Proof: $P: \{x \mid A\vec{x} \leq b\}$.



Let $v = \vec{c}^T \vec{x}$ be the optimal value of $\vec{c}^T \vec{x}$, $\vec{x} \in P$.

Let Q be the set of all optimal solutions ie.

$$Q: \{\vec{x} \mid A\vec{x} \leq b, \ \vec{c}^T \vec{x} = v\}.$$

Q is also a polyhedron. $Q \subset P$, and P has no lines. $\Rightarrow$ Q has no lines.

Q also has an extreme point.

Let $\vec{x}^*$ be an extreme point of Q. We will show that $\vec{x}^*$ is also an extreme point of P.

If possible, $\vec{x}^*$ is not an extreme point of P.

Then: $\exists \ \vec{y}, \vec{z} \in P$, $\vec{y} \neq \vec{x}, \vec{z} \neq \vec{x}^*$ s.t.

$$\vec{x}^* = \lambda \cdot \vec{y} + (1-\lambda) \vec{z}.$$

Now,

$$\vec{c}^T \vec{x}^* = v.$$

$$\Rightarrow \vec{c}^T \cdot \lambda \vec{y} + \vec{c}^T \cdot (1-\lambda) \cdot \vec{z}) = v$$

But $\vec{v}$ is the optimal cost.

$$\Rightarrow \quad \vec{c}^T \vec{y} \geq v \qquad \text{and} \qquad \vec{c}^T \vec{z} \geq v$$

$$\Rightarrow \quad \vec{c}^T \vec{y} = \vec{c}^T \vec{z} = v.$$

$$\Rightarrow \quad \vec{y} \in Q, \ \vec{z} \in Q.$$

$\Rightarrow$ $\vec{x}^*$ cannot be an extreme point of Q.

\# contradiction! Our assumption was wrong.

$\Rightarrow$ $\vec{x}^*$ must be an extreme point of P.

$\therefore$ An extreme point of P is optimal

QED

→ Foundations of the Simplex algorithm.