Optimization Models

EECS 127 / EECS 227AT

Laurent El Ghaoui

EECS department
UC Berkeley

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LECTURE 14

Review: Conic Models

We find in the history of ideas mutations which do not seem to correspond to any obvious need, and at first sight appear as mere playful whimsies such as Apollonius' work on conic sections, or the non-Euclidean geometries, whose practical value became apparent only later.

Arthur Koestler

Outline

Linear algebra

What we have seen in linear algebra:

- Matrix-vector, matrix-matrix product.
- Norms, projection on a line.
- Solving linear equations, least-squares.
- Spectral decomposition of symmetric matrices.
- Singular value decomposition of arbitrary matrices.
- Principal component analysis, low-rank approximation.

Linear algebra problems as optimization problems

- Solving linear systems of equations: (A is a matrix, b a vector)

$$\min_x \; 0 \; : \; Ax = b$$

- Ridge regression: (X is a data matrix, y a vector and $\lambda > 0$ a "regularization" parameter)

$$\min_w \; \|X^T w - y\|_2^2 + \lambda \|w\|_2^2$$

  (includes projection on a line as special case!);
- Maximum-variance direction and PCA ($C = C^T \succeq 0$ is a covariance matrix):

$$\max_w \; w^T C w \; : \; w^T w = 1,$$

- Low-rank approximation:

$$\min_{p,q} \; \|X - xy^T\|_F$$

  where x, y are vectors, and X is a given data matrix. Norm in objective can be the largest singular value norm, with same result.

Conic optimization

Linear programming

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.:} \quad & Ax \leq b, \quad Cx = d \end{aligned}$$

,

with A, b, c, C, d matrices or vectors of appropriate size.

Important applications examples:

- solving linear equations;
- linear regression problems based on l_1- and l_∞-norms;
- resource management;
- network flows.

Conic optimization

Quadratic programming

$$\begin{array}{rl} \text{minimize} & x^\top Q x + c^\top x \\ \text{subject to:} & Ax \le b, \quad Cx = d, \end{array}$$

with A, b, C, d, H matrices or vectors of appropriate size, and with Q PSD (symmetric and positive-semidefinite, also denoted $Q \succeq 0$). When $Q \succ 0$ (positive-definite), QP is a regularized version of LP, with a unique solution (if problem is feasible). The model includes LP as a special case.

Important applications examples:

- solving linear equations via least-squares;
- sparsity-constrained least-squares (LASSO);
- portfolio optimization, index tracking;
- linear-quadratic control.

Conic optimization

Quadratically constrained quadratic programming

The convex quadratic-constrained quadratic program (QCQP) model is

$$
\begin{aligned}
\min_{x} \quad & x^\top Q_0 x + a_0^\top x \\
\text{s.t.:} \quad & x^\top Q_i x + a_i^\top x \le b_i, \quad i = 1, \dots, m,
\end{aligned}
$$

with a_i, b_i vectors and scalars, and PSD matrices Q_i PSD ($Q_i = Q_i^\top \succeq 0$), $i = 0, 1, \dots, m$. The model includes LP, QP as a special case.

Important applications examples:

- minimization of the maximum of quadratic functions;
- Geometric problems, such as finding a point in the intersection of ellipses.
- portfolio optimization with multiple risk (variance) constraints.

Conic optimization

Second-order cone programming

The SOCP model is

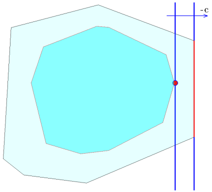
$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^\top x \\ \text{s.t.:} \quad & \|A_i x + b_i\|_2 \le c_i^\top x + d_i, \quad i = 1, \dots, m, \end{aligned}$$

with $A_i, b_i, c_i, d_i,\ i = 1, \dots, m$ matrices of appropriate size. The model includes LP, QP and QCQP as a special case.

Important applications examples:

- linear regression problems involving sums-of-powers of variables;
- grasping problems in robotics; truss optimization; etc.
- robust linear programming with ellipsoidal uncertainty.

Robust optimization



Robust counterpart:

$$
\begin{array}{ll}
\min\limits_{x} & \max\limits_{u \in \mathcal{U}} f_0(x,u) \\
\text{s.t.} & \forall\, u \in \mathcal{U}, \quad f_i(x,u) \le 0, \\
 & i = 1, \ldots, m
\end{array}
$$

- functions f_i now depend on a second variable u, the "uncertainty", which is constrained to lie in given set $\mathcal{U}$;
- in the case of robust linear programming, and with sets $\mathcal{U}$ that are ellipsoids or boxes, robust counterpart is tractable (an SOCP).

Are there other conic problems?

The last "Russian doll"

Yes!

- It's called "semidefinite programming" (SDP), and it involves optimization with PSD matrices as variables.
- It includes SOCP as a special case.
- SDPs are beyond the scope of this class, but are extensively covered in EE 227BT.

Solving conic problems

Due to their structure, conic problems can be efficiently solved globally:

- conic optimization solvers can provide a (near) optimal point, or unambiguously determine that the problem is infeasible;
- in practical terms, general SOCPs with dense input data and tens of thousands of variables and constraints can be solved in minutes on an ordinary laptop;
- With sparse input data the reach is much higher.

This is in sharp contrast with solvers for general nonlinear programming:

- users must provide a "good" initial guess;
- a "wrong" one may lead to a very sub-optimal solution;
- in the case of constrained problems, the solver may fail to find a feasible (let alone optimal) point, resulting in total failure, with little in the way of diagnostics.

Beyond conic optimization?

It turns out that one can extend some of the nice properties of conic models to a much broader class called "convex optimization".

- It is not true that every convex problem is easy to solve numerically, but "most" of them are;
- understanding the precise boundaries of what is easy and what is not is actually tricky;
- in the remainder of the class we will focus on the theoretical properties of convex programs;
- in particular, on a very important concept called duality.