Optimization Models

EECS 127 / EECS 227AT

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LECTURE 2

Vectors and Functions

Mathematicians are like Frenchmen: whatever you say to them, they translate into their own language, and turn it into something entirely different.

Goethe

Outline

Introduction

- A vector is a collection of numbers, arranged in a column or a row, which can be thought of as the coordinates of a point in n-dimensional space.
- Equipping vectors with sum and scalar multiplication allows to define notions such as independence, span, subspaces, and dimension. Further, the *scalar product* introduces a notion of angle between two vectors, and induces the concept of length, or norm.
- Via the scalar product, we can also view a vector as a linear function. We can compute the projection of a vector onto a line defined by another vector, onto a plane, or more generally onto a subspace.
- Projections can be viewed as a first elementary optimization problem (finding the point in a given set at minimum distance from a given point), and they constitute a basic ingredient in many processing and visualization techniques for high-dimensional data.

Basics

Notation

- We usually write vectors in column format:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

  Element x_i is said to be the i-th component (or the i-th element, or entry) of vector x, and the number n of components is usually referred to as the *dimension* of x.

- When the components of x are real numbers, i.e. $x_i \in \mathbb{R}$, then x is a real vector of dimension n, which we indicate with the notation $x \in \mathbb{R}^n$.

- We shall seldom need *complex* vectors, which are collections of complex numbers $x_i \in \mathbb{C}$, $i = 1, \ldots, n$. We denote the set of such vectors by $\mathbb{C}^n$.

- To transform a column-vector x in row format and vice versa, we define an operation called *transpose*, denoted with a superscript $^\top$:

$$x^\top = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}; \quad x^{\top\top} = x.$$

Examples

Example 1 (Bag-of-words representations of text)

Consider the following text:

> *"A (real) vector is just a collection of real numbers, referred to as the components (or, elements) of the vector; $\mathbb{R}^n$ denotes the set of all vectors with n elements. If $x \in \mathbb{R}^n$ denotes a vector, we use subscripts to denote elements, so that x_i is the i-th component of x. Vectors are arranged in a column, or a row. If x is a column vector, $x^\top$ denotes the corresponding row vector, and vice-versa."*

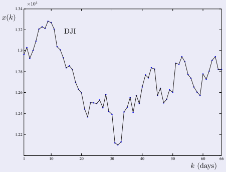
- *Row vector* $c = [5,\ 3,\ 3,\ 4]$ *contains the number of times each word in the list* $V = \{\textit{vector}, \textit{elements}, \textit{of}, \textit{the}\}$ *appears in the above paragraph.*
- *Dividing each entry in c by the total number of occurrences of words in the list (15, in this example), we obtain a vector* $x = [1/3, 1/5, 1/5, 4/15]$ *of relative word frequencies.*
- *Frequency-based representation of text documents (bag-of-words).*

Examples

Example 2 (Time series)

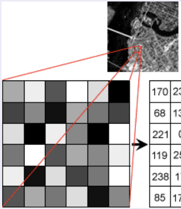
- *A* time series *represents the evolution in (discrete) time of a physical or economical quantity.*
- *If $x(k)$, $k = 1, \ldots, T$, describes the numerical value of the quantity of interest at time k, then the whole time series, over the time horizon from 1 to T, can be represented as a T-dimensional vector x containing all the values of $x(k)$, for $k = 1$ to $k = T$, that is*
$$x = [x(1)\ x(2)\ \cdots\ x(T)]^\top \in \mathbb{R}^T.$$

Adjusted close price of the Dow Jones Industrial Average Index, over a 66 days period from April 19, 2012 to July 20, 2012.



Example 3 (Images)

We are given a gray-scale image where each pixel has a certain value representing the luminance level (0=black). We can arrange the image as a vector of pixels.



170	238	85	255	221	0
68	136	17	170	119	68
221	0	238	136	0	255
119	255	85	170	136	238
238	17	221	68	119	255
85	170	119	221	17	136

170	238	85	255	221	0	. . .	85	170	119	221	17	136

Figure: *Row vector representation of an image.*

Vector spaces

- The operations of sum, difference and scalar multiplication are defined in an obvious way for vectors: for any two vectors $v^{(1)}, v^{(2)}$ having equal number of elements, we have that the sum $v^{(1)} + v^{(2)}$ is simply a vector having as components the sum of the corresponding components of the addends, and the same holds for the difference.
- If v is a vector and α is a scalar (i.e., a real or complex number), then αv is obtained multiplying each component of v by α. If $\alpha = 0$, then αv is the *zero vector*, or *origin*.
- A *vector space*, $\mathcal{X}$, is obtained by equipping vectors with the operations of addition and multiplication by a scalar.
- A simple example of a vector space is $\mathcal{X} = \mathbb{R}^n$, the space of n-tuples of real numbers. A less obvious example is the set of single-variable polynomials of a given degree.

Subspaces and span

- A nonempty subset $\mathcal{V}$ of a vector space $\mathcal{X}$ is called a *subspace* of $\mathcal{X}$ if, for any scalars α, β,
$$x, y \in \mathcal{V} \;\Rightarrow\; \alpha x + \beta y \in \mathcal{V}.$$
In other words, $\mathcal{V}$ is "closed" under addition and scalar multiplication.
- A *linear combination* of a set of vectors $S = \{x^{(1)}, \ldots, x^{(m)}\}$ in a vector space $\mathcal{X}$ is a vector of the form $\alpha_1 x^{(1)} + \cdots + \alpha_m x^{(m)}$, where $\alpha_1, \ldots, \alpha_m$ are given scalars.
- The set of all possible linear combinations of the vectors in $S = \{x^{(1)}, \ldots, x^{(m)}\}$ forms a subspace, which is called the subspace generated by S, or the *span* of S, denoted with $\mathrm{span}(S)$.
- Given two subspaces $\mathcal{X}, \mathcal{Y}$ in $\mathbb{R}^n$, the direct sum of $\mathcal{X}, \mathcal{Y}$, which we denote by $\mathcal{X} \oplus \mathcal{Y}$, is the set of vectors of the form $x + y$, with $x \in \mathcal{X}$, $y \in \mathcal{Y}$. It is readily checked that $\mathcal{X} \oplus \mathcal{Y}$ is itself a subspace.

Bases and dimensions

- A collection $x^{(1)}, \ldots, x^{(m)}$ of vectors in a vector space $\mathcal{X}$ is said to be *linearly independent* if no vector in the collection can be expressed as a linear combination of the others. This is the same as the condition

$$\sum_{i=1}^{m} \alpha_i x^{(i)} = 0 \Longrightarrow \alpha = 0.$$

- Given a subspace $\mathcal{S}$ of a vector space $\mathcal{X}$, a basis of $\mathcal{S}$ is a set B of vectors of minimal cardinality, such that $\mathrm{span}(B) = \mathcal{S}$. The cardinality of a basis is called the *dimension* of $\mathcal{S}$.
- If we have a basis $\{x^{(1)}, \ldots, x^{(d)}\}$ for a subspace $\mathcal{S}$, then we can write any element in the subspace as a linear combination of elements in the basis. That is, any $x \in \mathcal{S}$ can be written as

$$x = \sum_{i=1}^{d} \alpha_i x^{(i)},$$

for appropriate scalars α_i

Affine sets

- An affine set is a set of the form

$$\mathcal{A} = \{x \in \mathcal{X} : x = v + x^{(0)}, \, v \in \mathcal{V}\},$$

  where $x^{(0)}$ is a given point and $\mathcal{V}$ is a given subspace of $\mathcal{X}$. Subspaces are just affine spaces containing the origin.

- Geometrically, an affine set is a flat passing through $x^{(0)}$. The dimension of an affine set $\mathcal{A}$ is defined as the dimension of its generating subspace $\mathcal{V}$.

- A *line* is a one-dimensional affine set. The line through x_0 along direction u is the set

$$L = \{x \in \mathcal{X} : x = x_0 + v, \; v \in \mathrm{span}(u)\},$$

  where in this case $\mathrm{span}(u) = \{\lambda u : \lambda \in \mathbb{R}\}$.

Euclidean length

- The Euclidean length of a vector $x \in \mathbb{R}^n$ is the square-root of the sum of squares of the components of x, that is

$$\text{Euclidean length of } x \doteq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

  This formula is an obvious extension to the multidimensional case of the Pythagoras theorem in $\mathbb{R}^2$.
- The Euclidean length represents the actual distance to be "travelled" for reaching point x from the origin 0, along the most direct way (the straight line passing through 0 and x).

Basics

Norms and ℓ_p norms

- A *norm* on a vector space $\mathcal{X}$ is a real-valued function with special properties that maps any element $x \in \mathcal{X}$ into a real number $\|x\|$.

Definition 1

A function from $\mathcal{X}$ to $\mathbb{R}$ is a norm, *if*

$$\begin{aligned} &\|x\| \geq 0 \; \forall x \in \mathcal{X}, \text{ and } \|x\| = 0 \text{ if and only if } x = 0; \\ &\|x + y\| \leq \|x\| + \|y\|, \text{ for any } x, y \in \mathcal{X} \text{ (triangle inequality)}; \\ &\|\alpha x\| = |\alpha| \|x\|, \text{ for any scalar } \alpha \text{ and any } x \in \mathcal{X}. \end{aligned}$$

- ℓ_p norms are defined as

$$\|x\|_p \doteq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}, \quad 1 \leq p < \infty.$$

Basics

Norms and ℓ_p norms

- For $p = 2$ we obtain the standard Euclidean length

$$\|x\|_2 \doteq \sqrt{\sum_{k=1}^{n} x_k^2},$$

- or $p = 1$ we obtain the sum-of-absolute-values length

$$\|x\|_1 \doteq \sum_{k=1}^{n} |x_k|.$$

- The limit case $p = \infty$ defines the ℓ_∞ norm (max absolute value norm, or Chebyshev norm)

$$\|x\|_\infty \doteq \max_{k=1,\ldots,n} |x_k|.$$

- The cardinality of a vector x is often called the ℓ_0 (pseudo) norm and denoted with $\|x\|_0$.

Inner product

- An *inner product* on a (real) vector space $\mathcal{X}$ is a real-valued function which maps any pair of elements $x, y \in \mathcal{X}$ into a scalar denoted as $\langle x, y \rangle$. The inner product satisfies the following axioms: for any $x, y, z \in \mathcal{X}$ and scalar α

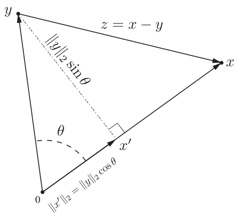
$$
\begin{aligned}
&\langle x, x \rangle \geq 0; \\
&\langle x, x \rangle = 0 \text{ if and only if } x = 0; \\
&\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle; \\
&\langle \alpha x, y \rangle = \alpha \langle x, y \rangle; \\
&\langle x, y \rangle = \langle y, x \rangle.
\end{aligned}
$$

- A vector space equipped with an inner product is called an *inner product space*.
- The *standard inner product* defined in $\mathbb{R}^n$ is the "row-column" product of two vectors

$$\langle x, y \rangle = x^\top y = \sum_{k=1}^{n} x_k y_k.$$

- The inner product induces a norm: $\|x\| = \sqrt{\langle x, x \rangle}$.

Angle between vectors



- The angle between x and y is defined via the relation

$$\cos\theta = \frac{x^\top y}{\|x\|_2 \|y\|_2}.$$

- When $x^\top y = 0$, the angle between x and y is $\theta = \pm 90^\circ$, i.e., x, y are *orthogonal*.
- When the angle θ is 0°, or $\pm 180^\circ$, then x is aligned with y, that is $y = \alpha x$, for some scalar α, i.e., x and y are *parallel*. In this situation $|x^\top y|$ achieves its maximum value $|\alpha| \|x\|_2^2$.

Cauchy-Schwartz and Hölder inequality

- Since $|\cos\theta| \leq 1$, it follows from the angle equation that

$$|x^\top y| \leq \|x\|_2 \|y\|_2,$$

and this inequality is known as the *Cauchy-Schwartz* inequality.

- A generalization of this inequality involves general ℓ_p norms and it is known as the *Hölder inequality*.
- For any vectors $x, y \in \mathbb{R}^n$ and for any $p, q \geq 1$ such that $1/p + 1/q = 1$, it holds that

$$|x^\top y| \leq \sum_{k=1}^{n} |x_k y_k| \leq \|x\|_p \|y\|_q.$$

Maximization of inner product over norm balls

- Our first optimization problem:
$$\max_{\|x\|_p \le 1} x^\top y.$$
- For $p = 2$:
$$x_2^* = \frac{y}{\|y\|_2},$$
hence $\max_{\|x\|_2 \le 1} x^\top y = \|y\|_2$.
- For $p = \infty$:
$$x_\infty^* = \mathrm{sgn}(y),$$
and $\max_{\|x\|_\infty \le 1} x^\top y = \sum_{i=1}^n |y_i| = \|y\|_1$.
- For $p = 1$:
$$[x_1^*]_i = \begin{cases} \mathrm{sgn}(y_i) & \text{if } i = m \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, \dots, n,$$
where m is an index such that $|y_i| \le |y_m|$ for all i. We thus have $\max_{\|x\|_1 \le 1} x^\top y = \max_i |y_i| = \|y\|_\infty$.

Orthogonal vectors

- Generalizing the concept of orthogonality to generic inner product spaces, we say that two vectors x, y in an inner product space $\mathcal{X}$ are *orthogonal* if $\langle x, y \rangle = 0$. Orthogonality of two vectors $x, y \in \mathcal{X}$ is symbolized by $x \perp y$.
- Nonzero vectors $x^{(1)}, \ldots, x^{(d)}$ are said to be *mutually orthogonal* if $\langle x^{(i)}, x^{(j)} \rangle = 0$ whenever $i \neq j$. In words, each vector is orthogonal to all other vectors in the collection.

Proposition 1

Mutually orthogonal vectors are linearly independent.

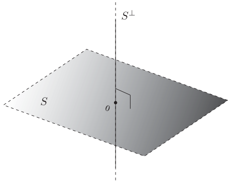
- A collection of vectors $S = \{x^{(1)}, \ldots, x^{(d)}\}$ is said to be *orthonormal* if, for $i, j = 1, \ldots, d$,

$$\langle x^{(i)}, x^{(j)} \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

In words, S is orthonormal if every element has unit norm, and all elements are orthogonal to each other. A collection of orthonormal vectors S forms an *orthonormal basis* for the span of S.

Orthogonal complement

- A vector $x \in \mathcal{X}$ is orthogonal to a subset $\mathcal{S}$ of an inner product space $\mathcal{X}$ if $x \perp s$ for all $s \in \mathcal{S}$.
- The set of vectors in $\mathcal{X}$ that are orthogonal to $\mathcal{S}$ is called the *orthogonal complement* of $\mathcal{S}$, and it is denoted with $\mathcal{S}^\perp$;



Theorem 1 (Orthogonal decomposition)

If $\mathcal{S}$ is a subspace of an inner-product space $\mathcal{X}$, then any vector $x \in \mathcal{X}$ can be written in a unique way as the sum of an element in $\mathcal{S}$ and one in the orthogonal complement $\mathcal{S}^\perp$:

$$\mathcal{X} = \mathcal{S} \oplus \mathcal{S}^\perp \quad \text{for any subspace } \mathcal{S} \subseteq \mathcal{X}.$$

Projections

- The idea of projection is central in optimization, and it corresponds to the problem of finding a point on a given set that is closest (in norm) to a given point.
- Given a vector x in an inner product space $\mathcal{X}$ (say, e.g., $\mathcal{X} = \mathbb{R}^n$) and a closed set $\mathcal{S} \subseteq \mathcal{X}$, the projection of x onto $\mathcal{S}$, denoted as $\Pi_{\mathcal{S}}(x)$, is defined as the point in $\mathcal{S}$ at minimal distance from x:

$$\Pi_{\mathcal{S}}(x) = \arg\min_{y \in \mathcal{S}} \|y - x\|,$$

  where the norm used here is the norm induced by the inner product, that is $\|y - x\| = \sqrt{\langle y - x, y - x \rangle}$.
- This simply reduces to the Euclidean norm, when using the standard inner product, in which case the projection is called *Euclidean projection*.

Projections

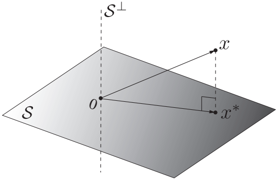
Theorem 2 (Projection Theorem)

Let $\mathcal{X}$ be an inner product space, let x be a given element in $\mathcal{X}$, and let $\mathcal{S}$ be a subspace of $\mathcal{X}$. Then, there exists a unique vector $x^* \in \mathcal{S}$ which is solution to the problem

$$\min_{y \in \mathcal{S}} \|y - x\|.$$

Moreover, a necessary and sufficient condition for x^* being the optimal solution for this problem is that

$$x^* \in \mathcal{S}, \quad (x - x^*) \perp \mathcal{S}.$$



Projections

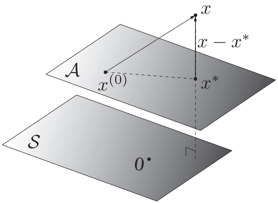
Corollary 1 (Projection on affine set)

Let $\mathcal{X}$ be an inner product space, let x be a given element in $\mathcal{X}$, and let $\mathcal{A} = x^{(0)} + \mathcal{S}$ be the affine set obtained by translating a given subspace $\mathcal{S}$ by a given vector $x^{(0)}$. Then, there exists a unique vector $x^* \in \mathcal{A}$ which is solution to the problem

$$\min_{y \in \mathcal{A}} \|y - x\|.$$

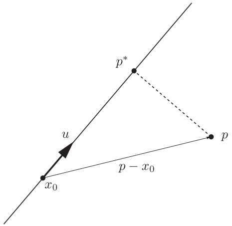
Moreover, a necessary and sufficient condition for x^* to be the optimal solution for this problem is that

$$x^* \in \mathcal{A}, \quad (x - x^*) \perp \mathcal{S}.$$



Projections

Euclidean projection of a point onto a line



- Let $p \in \mathbb{R}^n$ be a given point. We want to compute the Euclidean projection p^* of p onto a line $L = \{x_0 + \mathrm{span}(u)\}$, $\|u\|_2 = 1$:

$$p^* = \arg\min_{x \in L} \|x - p\|_2.$$

- Since any point $x \in L$ can be written as $x = x_0 + v$, for some $v \in \mathrm{span}(u)$, the above problem is equivalent to finding a value v^* for v, such that

$$v^* = \arg\min_{v \in \mathrm{span}(u)} \|v - (p - x_0)\|_2.$$

Projections

Euclidean projection of a point onto a line

- The solution must satisfy the orthogonality condition $(z - v^*) \perp u$. Recalling that $v^* = \lambda^* u$ and $u^\top u = \|u\|_2^2 = 1$, we hence have

$$u^\top z - u^\top v^* = 0 \iff u^\top z - \lambda^* = 0 \iff \lambda^* = u^\top z = u^\top (p - x_0).$$

- The optimal point p^* is thus given by

$$p^* = x_0 + v^* = x_0 + \lambda^* u = x_0 + u^\top (p - x_0) u,$$

- The squared distance from p to the line is

$$\|p - p^*\|_2^2 = \|p - x_0\|_2^2 - \lambda^{*2} = \|p - x_0\|_2^2 - (u^\top (p - x_0))^2.$$

Projections

Euclidean projection of a point onto an hyperplane

- A hyperplane is an affine set defined as

$$H = \{z \in \mathbb{R}^n : a^\top z = b\},$$

  where $a \neq 0$ is called a *normal direction* of the hyperplane, since for any two vectors $z_1, z_2 \in H$ it holds that $(z_1 - z_2) \perp a$.
- Given $p \in \mathbb{R}^n$ we want to determine the Euclidean projection p^* of p onto H.
- The projection theorem requires $p - p^*$ to be orthogonal to H. Since a is a direction orthogonal to H, the condition $(p - p^*) \perp H$ is equivalent to saying that $p - p^* = \alpha a$, for some $\alpha \in \mathbb{R}$.

Projections

Euclidean projection of a point onto an hyperplane

- To find α, consider that $p^* \in H$, thus $a^\top p^* = b$, then consider the optimality condition

  $$p - p^* = \alpha a$$

  and multiply it on the left by $a^\top$, obtaining

  $$a^\top p - b = \alpha \|a\|_2^2$$

  whereby

  $$\alpha = \frac{a^\top p - b}{\|a\|_2^2},$$

  and

  $$p^* = p - \frac{a^\top p - b}{\|a\|_2^2} a.$$

- The distance from p to H is

  $$\|p - p^*\|_2 = |\alpha| \cdot \|a\|_2 = \frac{|a^\top p - b|}{\|a\|_2}.$$

Projections

Projection on a vector span

- Suppose we have a basis for a subspace $\mathcal{S} \subseteq \mathcal{X}$, that is

$$\mathcal{S} = \mathrm{span}(x^{(1)}, \ldots, x^{(d)}).$$

- Given $x \in \mathcal{X}$, the Projection Theorem states that the unique projection x^* of x onto $\mathcal{S}$ is characterized by $(x - x^*) \perp \mathcal{S}$.
- Since $x^* \in \mathcal{S}$, we can write x^* as some (unknown) linear combination of the elements in the basis of $\mathcal{S}$, that is

$$x^* = \sum_{i=1}^{d} \alpha_i x^{(i)}.$$

  Then $(x - x^*) \perp \mathcal{S} \Leftrightarrow \langle x - x^*, x^{(k)} \rangle = 0,\ k = 1, \ldots, d$:

$$\sum_{i=1}^{d} \alpha_i \langle x^{(k)}, x^{(i)} \rangle = \langle x^{(k)}, x \rangle, \quad k = 1, \ldots, d.$$

- Solving this system of linear equations (aka the Gram equations) provides the coefficients α, and hence the desired x^*.

Projections

Projection onto the span of orthonormal vectors

- If we have an orthonormal basis for a subspace $\mathcal{S} = \mathrm{span}(S)$, then it is immediate to obtain the projection x^* of x onto that subspace.
- This is due to the fact that, in this case, the Gram system of equations immediately gives the coefficients

$$\alpha_k = \langle x^{(k)}, x \rangle, \quad i = 1, \ldots, d.$$

- Therefore, we have that

$$x^* = \sum_{i=1}^{d} \langle x^{(i)}, x \rangle x^{(i)}.$$

- Given a basis $S = \{x^{(1)}, \ldots, x^{(d)}\}$ for a subspace $\mathcal{S} = \mathrm{span}(S)$, there are numerical procedures to construct an orthonormal basis for the same subspace (e.g., the Gram-Schmidt procedure and QR factorization).

Functions and maps

- A *function* takes a vector argument in $\mathbb{R}^n$, and returns a unique value in $\mathbb{R}$.
- We use the notation
$$f : \mathbb{R}^n \to \mathbb{R},$$
to refer to a function with "input" space $\mathbb{R}^n$. The "output" space for functions is $\mathbb{R}$.
- For example, the function $f : \mathbb{R}^2 \to \mathbb{R}$ with values
$$f(x) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$
gives the Euclidean distance from the point (x_1, x_2) to a given point (y_1, y_2).
- We allow functions to take infinity values. The *domain* of a function f, denoted $\mathrm{dom}\, f$, is defined as the set of points where the function is finite.

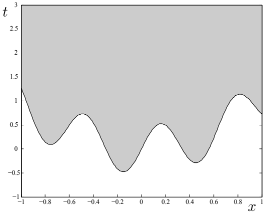
Functions and maps

- We usually reserve the term *map* to refer to vector-valued functions.
- That is, maps are functions that return more a vector of values. We use the notation

$$f : \mathbb{R}^n \to \mathbb{R}^m,$$

  to refer to a map with input space $\mathbb{R}^n$ and output space $\mathbb{R}^m$.
- The *components* of the map f are the (scalar-valued) functions f_i, $i = 1, \ldots, m$.

Sets related to functions



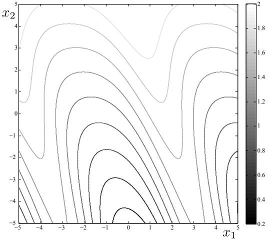
- Consider a function $f : \mathbb{R}^n \to \mathbb{R}$.
- The *graph* and the *epigraph* of a function $f : \mathbb{R}^n \to \mathbb{R}$ are both subsets of $\mathbb{R}^{n+1}$.
- The *graph* of f is the set of input-output pairs that f can attain, that is:

$$\operatorname{graph} f = \left\{ (x, f(x)) \in \mathbb{R}^{n+1} \ : \ x \in \mathbb{R}^n \right\}.$$

- The *epigraph*, denoted $\operatorname{epi} f$, describes the set of input-output pairs that f can achieve, as well as "anything above":

$$\operatorname{epi} f = \left\{ (x, t) \in \mathbb{R}^{n+1} \ : \ x \in \mathbb{R}^n, \ \ t \geq f(x) \right\}.$$

Sets related to functions



- A *level set* (or *contour* line) is the set of points that achieve exactly some value for the function f. For $t \in \mathbb{R}$, the t-level set of the function f is defined as

$$C_f(t) = \{x \in \mathbb{R}^n : \ f(x) = t\}\,.$$

- The t-sublevel set of f is the set of points that achieve at most a certain value for f:

$$L_f(t) = \{x \in \mathbb{R}^n : \ f(x) \leq t\}\,.$$

Linear and affine functions

- Linear functions are functions that preserve scaling and addition of the input argument.
- A function $f : \mathbb{R}^n \to \mathbb{R}$ is *linear* if and only if

$$\forall x \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{R}, f(\alpha x) = \alpha f(x);$$
$$\forall x_1, x_2 \in \mathbb{R}^n, f(x_1 + x_2) = f(x_1) + f(x_2).$$

- A function f is *affine* if and only if the function $\tilde{f}(x) = f(x) - f(0)$ is linear (affine $=$ linear $+$ constant).
- Consider the functions $f_1, f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}$ defined below:

$$\begin{aligned} f_1(x) &= 3.2x_1 + 2x_2, \\ f_2(x) &= 3.2x_1 + 2x_2 + 0.15, \\ f_3(x) &= 0.001x_2^2 + 2.3x_1 + 0.3x_2. \end{aligned}$$

The function f_1 is linear; f_2 is affine; f_3 is neither linear nor affine (f_3 is a quadratic function).

Linear and affine functions

- Linear or affine functions can be conveniently defined by means of the standard inner product.
- A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine if and only if it can be expressed as

$$f(x) = a^\top x + b,$$

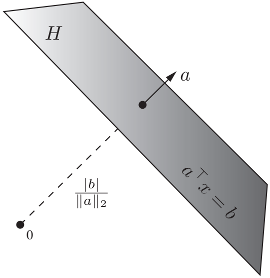
  for some unique pair (a, b), with a in $\mathbb{R}^n$ and $b \in \mathbb{R}$.
- The function is linear if and only if $b = 0$.
- Vector $a \in \mathbb{R}^n$ can thus be viewed as a (linear) map from the "input" space $\mathbb{R}^n$ to the "output" space $\mathbb{R}$.
- For any affine function f, we can obtain a and b as follows: $b = f(0)$, and $a_i = f(e_i) - b$, $i = 1, \ldots, n$.

Hyperplanes and halfspaces

- A hyperplane in $\mathbb{R}^n$ is a set of the form

$$H = \left\{ x \in \mathbb{R}^n \, : \, a^\top x = b \right\},$$

where $a \in \mathbb{R}^n$, $a \neq 0$, and $b \in \mathbb{R}$ are given.



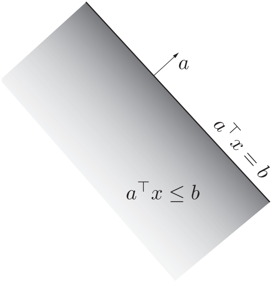
- Equivalently, we can think of hyperplanes as the level sets of linear functions.
- When $b = 0$, the hyperplane is simply the set of points that are orthogonal to a (i.e., H is a $(n-1)$-dimensional subspace).

Hyperplanes and halfspaces

- An hyperplane H separates the whole space in two regions:

$$H_{-} = \left\{x \,:\, a^\top x \leq b\right\}, \quad H_{++} = \left\{x \,:\, a^\top x > b\right\}.$$

- These regions are called halfspaces (H_{-} is a closed halfspace, H_{++} is an open halfspace).
- the halfspace H_{-} is the region delimited by the hyperplane $H = \{a^\top x = b\}$ and lying in the direction opposite to vector a. Similarly, the halfspace H_{++} is the region lying above (i.e., in the direction of a) the hyperplane.



Gradients

- The gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point x where f is differentiable, denoted with $\nabla f(x)$, is a column vector of first derivatives of f with respect to $x_1, \ldots, x_n$:

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \cdots & \frac{\partial f(x)}{\partial x_n} \end{bmatrix}^\top .$$

- When $n = 1$ (there is only one input variable), the gradient is simply the derivative.
- An affine function $f : \mathbb{R}^n \to \mathbb{R}$, represented as $f(x) = a^\top x + b$, has a very simple gradient: $\nabla f(x) = a$.

Example 4

The distance function $\rho(x) = \|x - p\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - p_i)^2}$ *has gradient*

$$\nabla \rho(x) = \frac{1}{\|x - p\|_2}(x - p).$$

Affine approximation of nonlinear functions

- A non-linear function $f : \mathbb{R}^n \to \mathbb{R}$ can be approximated locally via an affine function, using a first-order Taylor series expansion.
- Specifically, if f is differentiable at point x_0, then for all points x in a neighborhood of x_0, we have that

$$f(x) = f(x_0) + \nabla f(x_0)^\top (x - x_0) + \epsilon(x),$$

  where the error term $\epsilon(x)$ goes to zero faster than first order, as $x \to x_0$, that is

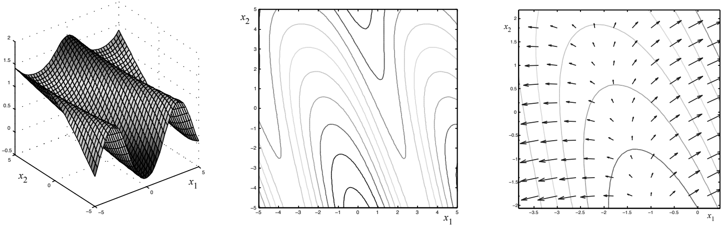
$$\lim_{x \to x_0} \frac{\epsilon(x)}{\|x - x_0\|_2} = 0.$$

- In practice, this means that for x sufficiently close to x_0, we can write the approximation

$$f(x) \simeq f(x_0) + \nabla f(x_0)^\top (x - x_0).$$

Geometric interpretation of the gradient

- The gradient of a function can be interpreted in the context of the level sets.
- Indeed, geometrically, the gradient of f at a point x_0 is a vector $\nabla f(x_0)$ perpendicular to the contour line of f at level $\alpha = f(x_0)$, pointing from x_0 outwards the α-sublevel set (that is, it points towards higher values of the function).



Geometric interpretation of the gradient

- The gradient $\nabla f(x_0)$ also represents the direction along which the function has the maximum rate of increase (steepest ascent direction).
- Let v be a unit direction vector (i.e., $\|v\|_2 = 1$), let $\epsilon \geq 0$, and consider moving away at distance ϵ from x_0 along direction v, that is, consider a point $x = x_0 + \epsilon v$. We have that

$$f(x_0 + \epsilon v) \simeq f(x_0) + \epsilon \nabla f(x_0)^\top v, \text{ for } \epsilon \to 0,$$

  or, equivalently,

$$\lim_{\epsilon \to 0} \frac{f(x_0 + \epsilon v) - f(x_0)}{\epsilon} = \nabla f(x_0)^\top v.$$

- Whenever $\epsilon > 0$ and v is such that $\nabla f(x_0)^\top v > 0$, then f is increasing along the direction v, for small ϵ.
- The inner product $\nabla f(x_0)^\top v$ measures the rate of variation of f at x_0, along direction v, and it is usually referred to as the *directional derivative* of f along v.

Geometric interpretation of the gradient

- The rate of variation is thus zero, if v is orthogonal to $\nabla f(x_0)$: along such a direction the function value remains constant (to first order), that is, this direction is tangent to the contour line of f at x_0.
- Contrary, the rate of variation is maximal when v is parallel to $\nabla f(x_0)$, hence along the normal direction to the contour line at x_0.