Lecture #24 Discrete Time State Space Approach (Nov. 14, 2018) v. 1.01
Ref: K. Ogata, *Discrete-Time Control Systems* 1995.
EE128, Fall 2015, R. Fearing

Consider the LTI system:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \qquad y = C\mathbf{x} + D\mathbf{u}. \tag{1}$$

An example of the behavior of an LTI system to a discrete time input is shown in Figure 1, where the control signal $u(t)$ is constant over the sample period T. (This corresponds to a *zero order hold* model.) We can predict the response to this input by looking at it as a superposition of step responses. The response at time kT is just

$$x(kT) = e^{AkT}x(0) + e^{AkT}\int_0^{kT} e^{-A\tau}Bu(\tau)d\tau \ . \tag{2}$$

It is important to note that eq.(2) gives the *exact* solution for $x(t = kT)$. At time T later the response is just:

$$x((k+1)T) = e^{A(k+1)T}x(0) + e^{A(k+1)T}\int_0^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau \ . \tag{3}$$

We can get a recursive relation for $x((k+1)T)$ by premultiplying eqn. 2 by e^{AT} and subtracting from eqn. 3, obtaining:

$$x((k+1)T) = e^{AT}x(kT) + e^{A(k+1)T}\int_{kT}^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau \ = e^{AT}x(kT) + \int_0^{T} e^{A\lambda}Bu(kT)d\lambda \ \ , \tag{4}$$

where for the last expression, we substituted $\lambda = (k+1)T - \tau$, and $d\lambda = -d\tau$.

Define two constants G and H which are a function only of the sample period T, where:

$$G(T) \equiv e^{AT} \qquad \text{and} \qquad H(T) \equiv \left(\int_0^T e^{A\lambda}d\lambda\right) B \ . \tag{5}$$

Now we can rewrite our continuous time system eqn. 1 in discrete time as:

$$x((k+1)T) = G(T)x(kT) + H(T)u(kT) \tag{6}$$

and

$$y(kT) = Cx(kT) + Du(kT) \ . \tag{7}$$

It is worthwhile to convince yourself that this is the exact solution for $x(t)$ when $t = kT$.

Discrete Time Solution

We can find an equivalent formula to eqn. 3 by explicitly solving the state equations as follows:

$$x(1) = Gx(0) + Hu(0) \tag{8}$$

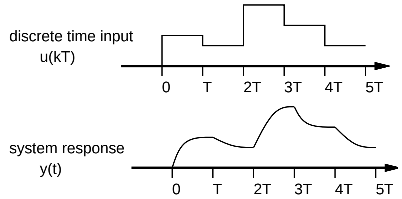

Figure 1: A discrete time control input with zero order hold applied to a continuous time system.

$$x(2) = Gx(1) + Hu(1) = G^2x(0) + GHu(0) + Hu(1) \tag{9}$$

$$x(3) = Gx(2) + Hu(2) = G^3x(0) + G^2Hu(0) + GHu(1) + Hu(2) \tag{10}$$

Thus we can write the discrete time convolution as

$$x(k) = G^k x(0) + \sum_{j=0}^{k-1} G^{k-j-1} Hu(j) \ . \tag{11}$$

For stability, the eigenvalues of G must have magnitude < 1. So if the continuous time plant is stable ($\Re$ eig $(A) < 0$) the discrete time plant will have eigenvalues with magnitude less than 1. As we see in the next subsection, a proportional feedback $u(kT) = k_p(r(kT) - y(kT))$ is not necessarily stable.

Discrete Time Controller

Let's see what happens to the overall system when a discrete time proportional control $u(kT) = k_p(r(kT) - x(kT))$ is added. Remember, our plant is still described by a linear differential equation (1) but now we are sensing the plant output at discrete time intervals, and changing the output at discrete intervals. We are neglecting any delay in computing the control law, which we might need to fix if our sample rate is low compared to system response frequencies. Thus our new discrete time state equations (6) become:

$$x((k+1)T) = G(T)x(kT) + H(T)k_p(r(kT) - x(kT)) = [G - Hk_p]x(kT) + Hk_pr(kT) \ . \tag{12}$$

Now consider a first order continuous time case, where the homogeneous solution could be made stable by making the proportional feedback k_p sufficiently positive and large. In discrete time, we need to be more careful. Let's see why for a first order example. Using eqn. 5 we get $H(T) = \int_0^T e^{a\lambda} d\lambda b = \frac{b}{a}(e^{aT} - 1)$, then

$$x((k+1)T) = [e^{aT} + \frac{k_p}{a}(1 - e^{aT})]x(kT) + Hk_pr(kT) \ = \ G'x(kT) + Hk_pr(kT) \ . \tag{13}$$

When is this controller stable? By inspection, the constant multiplying the $x(kT)$ term must have magnitude less than 1, or

$$|e^{aT} - \frac{k_p}{a}(e^{aT} - 1)| < 1. \tag{14}$$

If the original continuous time system is stable, then $a < 0$. Then if we look at $k_p = 0$ the system is obviously stable, since $e^{aT} < 1$. What happens as we try to increase k_p? We can increase k_p until $G' = 0$, then the response will be oscillatory for $-1 < G' < 0$, and unstable for $G' < -1$. Think about $G' = 0$ for a minute. If $G - Hk_p = 0$, this implies that you can control $x(kT)$ in one time step with no delay, the dynamics of the plant have disappeared.

Z Transform Solution

With initial condition $x[0]$ and assuming the existence of the *z transform* of $x(kT)$ and $u(kT)$ we have

$$zX(z) = GX(z) + HU(z) + x[0]z \tag{15}$$

and

$$X(z) = [zI - G]^{-1}(HU(z) + x[0]z) \quad . \tag{16}$$

References

[1] G. Franklin et al, *Feedback Control of Dynamic Systems* , 2nd edition

[2] B. Kuo, *Automatic Control Systems*, 6th edition

[3] K. Ogata, *Modern Control Engineering*, 2nd edition

[4] K. Ogata, *Discrete-Time Control Systems*, 2nd edition