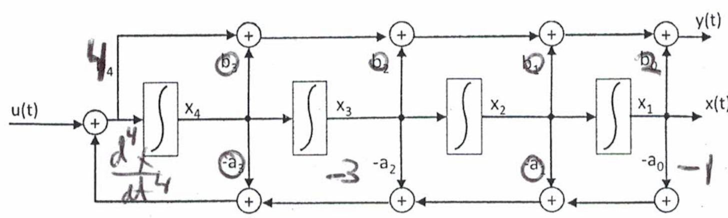
Key.

Problem 1 (25 pts)
Each part is independent.

[7 pts] a) Consider a single-input single-output system with input $u(t)$ and output $y(t)$ described by the block diagram below with coefficients as given. (Note that this diagram is a modified version of the Lec#3 handout.)

$a_0 = \underline{1}$ $a_1 = \underline{0}$ $a_2 = \underline{3}$ $a_3 = \underline{0}$
$b_0 = \underline{2}$ $b_1 = \underline{0}$ $b_2 = \underline{0}$ $b_3 = \underline{0}$ $b_4 = \underline{4}$

$x_2 = \dot{x}_1$
$x_3 = \dot{x}_2 = \ddot{x}$
$x_4 = \dot{x}_3 = \dddot{x}$
$\dot{x}_4 = \frac{d^4 x}{dt^4} = u - 3\ddot{x} - x$
$X(s)(s^4 + 3s^2 + 1) = U(s)$
$\frac{X(s)}{U(s)} = \frac{1}{s^4 + 3s^2 + 1}$
$Y(s) = (4s^4 + 2)X(s)$



[3 pts] i) Write the transfer function for the system:

$y = 4\frac{d^4 x}{dt^4} + 2x$
$= 4(u - 3\ddot{x} - x) + 2x$
$= 4u - 12x_3 - 2x_1$

$$\frac{Y(s)}{U(s)} = \frac{4s^4 + 2}{s^4 + 3s^2 + 1}$$

[4 pts] ii) For the output equation $y = Cx + Du = C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \cancel{b_4} 4 u(t)$, find C.

$C = [\underline{-2}\ \underline{0}\ \underline{-12}\ \underline{0}]$

[5 pts] b) A nonlinear system with output $x(t)$ and input $f(t)$ is described by the differential equation

$$\ddot{x} + \dot{x} + kx^2 = f(t)$$

The system input is a constant offset with small variations such that $f(t) = 1 + \delta f(t)$. The output $x(t) \approx x_o + \delta x(t)$. Find the transfer function relating output variation $\delta x(t)$ to input variation $\delta f(t)$.

$$\frac{\Delta X(s)}{\Delta F(s)} = \frac{1}{s^2 + s + 2kx_0}$$

ignore init conds

$\frac{d^2 x}{dt^2} = \frac{d}{dt^2}(x_0 + \delta x) = \ddot{\delta x}$

$\frac{dx}{dt} = \frac{d}{dt}(x_0 + \delta x) = \dot{\delta x}$

$kx^2 = k(x_0 + \delta x)^2 = k(x_0^2 + 2x_0\delta x + \delta x^2)$
assume δx small $\delta x^2 \approx 0$

$\ddot{\delta x} + \dot{\delta x} + k(x_0^2 + 2x_0\delta x) = 1 + \delta f(t)$

Variation: $\ddot{\delta x} + \dot{\delta x} + 2kx_0\delta x = \delta f(t)$
$s^2\Delta X(s) + s\Delta X(s) + 2kx_0 \Delta(x(s)) = \Delta F$

Key,

c) Consider a system with a **step** input which has output transfer function:

$$Y(s) = \frac{2(s-2)}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

[3 pts] i) Find the partial fraction expansion coefficients for $Y(s)$:

$A =$ -1

$B =$ 2

$C =$ -1

$$A = \frac{-4}{4} = -1$$

$$B = \frac{2\cdot(-3)}{(-1)(3)} = 2$$

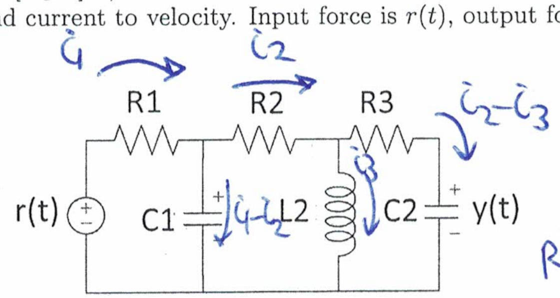
$$C = \frac{2(-6)}{(-4)(-3)} = -1$$

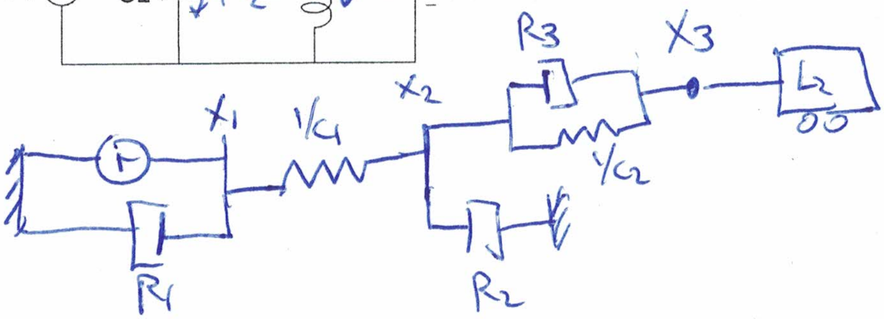
[2 pts] ii) Find $y(t)$, the inverse Laplace transform of $Y(s)$, using the partial fraction expansion above.

$y(t) =$

$$\left(-1 + 2e^{-t} - e^{-4t}\right)u(t).$$

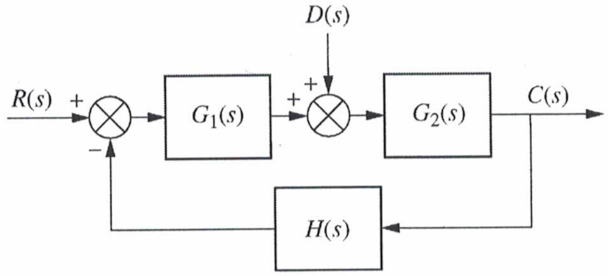
[8 pts] d) Draw the equivalent mechanical circuit for this electrical system, with voltage corresponding to force and current to velocity. Input force is $r(t)$, output force is y(t).





Key.

Problem 2 Steady State Error (22 pts)



For parts a) and b), use:

$G_1(s) = k$

$G_2(s) = \frac{s+2}{(s+3)(s+4)}$

$H(s) = \frac{1}{s}$

2' [4 pts] a) Let $R(s) = 0$. Find the response to a general disturbance input $D(s)$ in terms of $G_1(s), G_2(s), H(s)$.

$\frac{C(s)}{D(s)} = \frac{G_2}{1+G_1G_2H}$

$C = G_2(D + G_1(0 - HC))$

$C(1+G_1G_2H) = G_2D$

2' [4 pts] b) For a disturbance input $d(t) = u(t)$, a unit step, (with $r(t) = 0$) show that $\lim_{t\to\infty} c(t) = 0$.

$C(s) = \frac{G_2}{1+G_1G_2H} = \frac{(s+2)}{(s+3)(s+4) + \frac{k(s+2)}{s}}$, $\lim_{s\to 0} sC(s)\cdot\frac{1}{s} = \lim_{s\to 0} \frac{s(s+2)}{s(s+3)(s+4) + k(s+2)}$

$= 0.$

[4 pts] c) Let $D(s) = 0$. Let $e(t) = r(t) - c(t)$. Find $\frac{E(s)}{R(s)}$ in terms of $G_1(s), G_2(s), H(s)$.

$\frac{E(s)}{R(s)} = \frac{1 - G_1G_2 + G_1G_2H}{1+G_1G_2H}$

$C = G_1G_2(R - HC)$

$C(1+G_1G_2H) = G_1G_2R$

$\frac{C}{R} = \frac{G_1G_2}{1+G_1G_2H}$

$E = R - C$

$= R(1 - C/R)$

$= \frac{1+G_1G_2H - G_1G_2}{1+G_1G_2H}\cdot R$

[4 pts] d) For $r(t) = u(t)$, a unit step, (with $d(t) = 0$), show that $\lim_{t\to\infty} e(t) \neq 0$.

$\lim_{s\to 0} \cancel{s}\frac{E}{R}\cdot\frac{1}{\cancel{s}} = \lim_{s\to 0} \frac{1 + \frac{k(s+2)}{(s+3)(s+4)s} - \frac{k(s+2)}{(s+3)(s+4)}}{1 + \frac{k(s+2)}{(s+3)(s+4)s}}$

$\lim_{s\to 0} = \frac{(s+3)(s+4)s + k(s+2) - k(s+2)s}{(s+3)(s+4)s + k(s+2)}$

$= \frac{2k}{2k} = 1 = \lim_{t\to\infty} e(t)$

Key

Problem 3. Root Locus Plotting (27 pts)

For the root locus $(1 + kG(s) = 0)$ with $k > 0$, and given open loop transfer function $G(s)$:

$$G(s) = \frac{s(s+1)}{(s^2 - 2s + 5)(s^2 + 6s + 13)}$$

[1 pts] a) Determine the number of branches of the root locus = 4

[2 pts] b) Determine the locus of poles on the real axis $-1 < \sigma < 0$

[3 pts] c) Determine the angles for each asymptote: $\pi/2, -\pi/2$

$$\frac{(2l+1)\pi}{(4-2)} = \frac{\pi}{2}, -\pi/2$$

[4 pts] d) determine the real axis intercept for the asymptotes $\sigma =$

$$\frac{\sum \text{poles} - \sum \text{zeros}}{n-m} = \frac{1+1-3-3-0-(-1)}{2} = \frac{-3}{2}$$

[6 pts] e) Use the angle criteria for poles and zeros to show that $p \approx 0 + 2.1j$ is on the root locus.

zeros: $\angle p = \pi/2 = +90^\circ$ $\quad \angle(p+1) = +63^\circ$

poles:

$\angle(p - 1 + 2j) = -104^\circ$ $\quad \angle(p + 3 + 2j) = -53^\circ$

$\angle(p - 1 - 2j) = -\pi$ $\quad \angle(p + 3 - 2j) = 0$

-180°

zeros $+153^\circ$

poles $-180 - 157$

$\approx -180^\circ$.

[6 pts] f) Estimate the value of k for which $p \approx +2.1j$ is a closed-loop pole. (Show work for full credit).

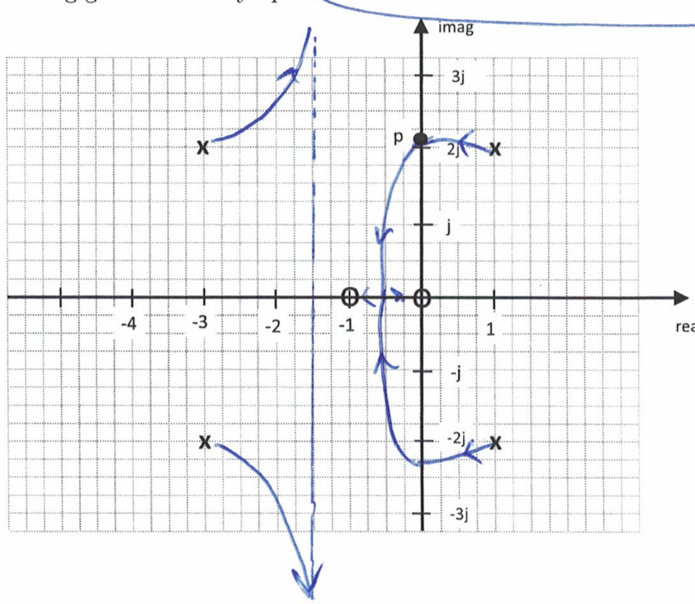
$k = 13.5$

$$kG = 1, \quad k = \frac{1}{|G|} = \frac{1 \cdot 3 \cdot \sqrt{17} \cdot \sqrt{25}}{2 \cdot \sqrt{5}\ \sqrt{17} \cdot \sqrt{25}} \approx \frac{3}{12} \cdot \frac{1}{\sqrt{85}} \approx \frac{1}{30}$$

$$= \frac{3}{2} \cdot \sqrt{5} \cdot \sqrt{17} \approx \frac{3}{2} \cdot \sqrt{81} \approx \frac{27}{2}$$

$\sqrt{85} \approx \sqrt{81} = 9$

[5 pts] g) Sketch the root locus below using the information found above. Draw arrows on branches showing increasing gain. Draw asymptotes.



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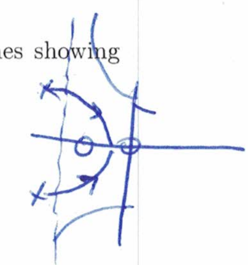
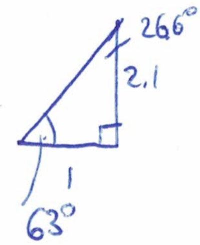
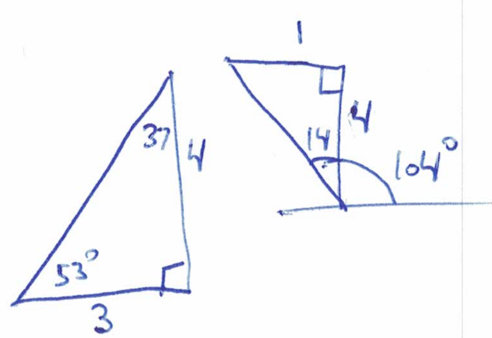
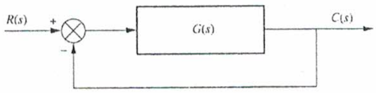






Key.

Problem 4. Root Locus Compensation (18 pts)



Given open loop transfer function $G(s)$, where $G_1(s)$ is the open-loop plant:

$$G(s) = G_c(s)G_1(s) = G_c(s)\frac{64}{(s^2+4s+8)^2}$$

and $G_c(s)$ is a PI compensation of the form $G_c(s) = k_p\frac{s+z_c}{s}$. The closed loop system, using unity gain feedback and the PI controller, should have a pair of poles at $p \approx -1+j$ and $p^* \approx -1-j$.

[7 pts] a. To obtain the closed loop pole at p, estimate the angle contribution (in degrees) for each of the open loop poles, and the total angle contribution for all open-loop poles:

pole	angle	pole	angle
s = 0	-135°		
s = -2+2j	+45	s = -2+2j	+45
s = -2-2j	−71.6	s = -2-2j	−71.6
TOTAL	-188.2°		

$-135+90 = -45^\circ$

-143.2°

-188.2

[2 pts] b. What is the necessary angle contribution of the zero z_c for the closed loop pole p to be on the root locus?

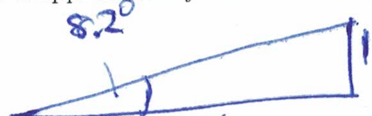
$+8.2^\circ$

$\pi \cdot \frac{8.2}{180} \approx 3 \cdot \frac{4}{90} \approx \frac{1}{10}\cdot\frac{4}{3} \approx 0.13$

[9 pts] c. Find z_c to within ±0.5 such that p is approximately on the root locus, within ± 2 degrees. (Show work.)

$z_c =$ $+1+7.5 = +8.5$

8.2°



$\tan^{-1} 1/8 = 7.1^\circ$

$\tan^{-1} 1/5 = 11.3^\circ$

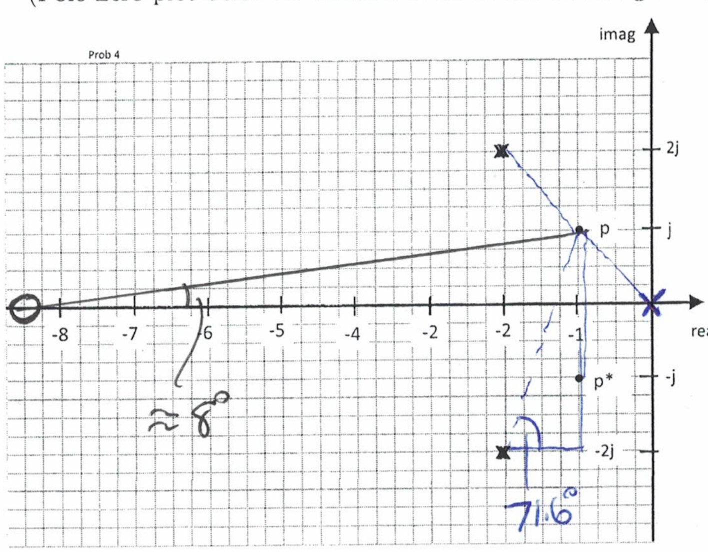
$\tan \frac{1}{x} \approx 0.13$

$z_c = +8.5$

$x \approx \frac{1}{0.13} \approx 7.5$

$G_c(s) = \frac{s+8.5}{s}$

(Pole-Zero plot below for scratch work. It will not be graded).



KEY

Problem 5. Routh-Hurwitz (14 pts)

Given system with closed loop transfer function (assuming unity feedback)

$$T(s) = \frac{k}{s^4 + 10s^3 + 33s^2 + 40s + (16 + k)}$$

[10 pts] a. Using the Routh-Hurwitz table, show that the maximum positive k for which the closed loop system is stable is approximately 100.

s^4	1	33	$16+k$	0
s^3	10	40	0	0
s^2	$\frac{33\cdot 10 - 40}{10} = 29$	$16+k$	0	
s^1	$\frac{29\cdot 40 - 10(16+k)}{29} = 40 - \frac{10}{29}(16+k)$	0	0	

$s^0 = 16+k$

>0 for $k > -16$

$40 - \frac{10}{29}(16+k) > 0,$

$40 > \frac{10}{29}(16+k)$

$4 \cdot 29 > 16+k$

$116 > 16+k$

$\boxed{100 > k}$

[4 pts] b. For $k \approx 100$, approximately find the pair of closed loop poles on the imaginary axis. (Show work).

$s = \pm j\omega_o = \pm j$ 2.

For $k=100$, s^1 row is zero,

s^2 row : $29s^2 + 116 = 0$

$s^2 + 4 = 0$

$s = \pm 2j$