Problem Set 4: Solutions

Problem 1

a. We have:

$$\begin{aligned}
C_1(s) &= \frac{8(s+2.5)}{s(s+2)(s^2+3s+10)} \\
&= \frac{8s+20}{s(s+2)(s^2+3s+10)} \\
&= \frac{A}{s} + \frac{B}{s+2} + \frac{Cs+D}{s^2+3s+10} \\
&= \frac{A(s+2)(s^2+3s+10) + Bs(s^2+3s+10) + (Cs+D)(s)(s+2)}{s(s+2)(s^2+3s+10)}
\end{aligned}$$

There are multiple ways to obtain the coefficients. The "official" way is to expand everything and compare coefficients of polynomials. An alternative, is to try "clever" values of s (usually the roots) to quickly obtain the coefficients solutions. Refer to Chapter 2 on the textbook for details on the partial fractions expansion.
Now comparing numerator, with $s \to 0$, it eliminates the B, C and D terms, and we have:

$$A \cdot (0+2) \cdot (0+10) = 8 \cdot 0 + 20 \to 20A = 20 \to A = 1$$

Similarly, for $s \to -2$ we have:

$$B \cdot (-2) \cdot (2^2 + 3 \cdot (-2) + 10) = 8 \cdot (-2) + 20 \to -16B = 4 \to B = -\frac{1}{4}$$

For the last case, C and D, there are no easy roots, so we will explore two techniques to obtain the coefficients.

- Similarly to the previous idea, you can just replace values that were not used. For example, for $s = 1$ we have (and replacing A and B):

$$\begin{aligned}
&A \cdot (1+2) \cdot (1^2+3+10) + B(1^2+3+10) + (C+D) \cdot 1 \cdot (1+2) = 8+20 \\
&42 + \frac{-1}{4} \cdot 14 + 3C + 3D = 28 \\
&3C + 3D = -\frac{21}{2}
\end{aligned}$$

  And for $s = -1$ we have:

$$\begin{aligned}
&A \cdot (-1+2) \cdot (1^2-3+10) + B(-1)(1^2-3+10) + (-C+D) \cdot (-1) \cdot (-1+2) = -8+20 \\
&8 + \frac{-1}{4} \cdot (-1) \cdot 8 + C - D = 12 \\
&C - D = 2
\end{aligned}$$

Solving the 2×2 system we obtain:

$$C = -\frac{3}{4} \qquad D = -\frac{11}{4}$$

- Now, the second alternative is the one described in the textbook based on matching coefficients via multiplying the lowest common denominator, $s(s+2)(s^2+3s+10)$:

$$8s + 20 = A(s+2)(s^2+3s+10) + Bs(s^2+3s+10) + (Cs+D)s(s+2)$$

Replacing A and B and expanding we have:

$$8s + 20 = \left(C + \frac{3}{4}\right)s^3 + \left(2C + D + \frac{17}{4}\right)s^2 + \left(2D + \frac{27}{2}\right)s + 20$$

Balancing coefficients between the polynomials, we have three equations for two variables. From the s^3 coefficient we have:

$$C + \frac{3}{4} = 0 \rightarrow C = -\frac{3}{4}$$

From the s coefficient we have:

$$2D + \frac{27}{2} = 8 \rightarrow D = -\frac{11}{4}$$

The same results as the previous technique. We confirm that the coefficient associated with s^2 is zero:

$$2C + D + \frac{17}{4} = -2 \cdot \frac{3}{4} - \frac{11}{4} + \frac{17}{4} = 0$$

as expected.

Finally, the partial fraction expansion is:

$$C_1(s) = \frac{1}{s} - \frac{1}{4(s+2)} + \frac{-3s-11}{4(s^2+3s+10)}$$

Observe that the residue (or coefficient) associated with the pole at $s = -2$ (related with the $(s+2)^{-1}$ term), which is closest to the zero at -2.5, is given by $-1/4$ and is NOT negligible in comparison to the other residues. This showcases that probably the pole-cancellation is not appropriate. We will verify this claim using the step response later.

b. Repeating the process from part a. we obtain:

$$C_2(s) = \frac{1.005}{s} - \frac{0.00625}{s+2} + \frac{-0.99875s - 3.002875}{s^2+3s+10}$$

Observe now that the residue of the pole at $s = -2$, which is closest to the zero at $s = -2.01$, is 0.00625 and seems to be negligible in comparison to the other residues (at least two order of magnitude below the other residues). So, this pole-cancellation seems to be appropriate.

The code used for MATLAB is as follows (note that $H_2(s)$ does not have a final value of 1, so we normalize by multiplying by 20/20.1 in the numerator):

Listing 1: MATLAB code that plots the step responses.

```
num1 = [8 20];
den1 = [1 5 16 20];
num1tilde = 10;
den1tilde = [1 3 10];
num2 = (20/20.1)*[10 20.1];

H1 = tf(num1, den1);
H1tilde = tf(num1tilde, den1tilde);
H2 = tf(num2, den1);

fig1 = figure();
step(H1)
hold on
step(H1tilde)
legend({'H1', 'H1tilde'})
hold off

fig2 = figure();
step(H2)
hold on
step(H1tilde)
legend({'H2', 'H2tilde'})
```

with responses:

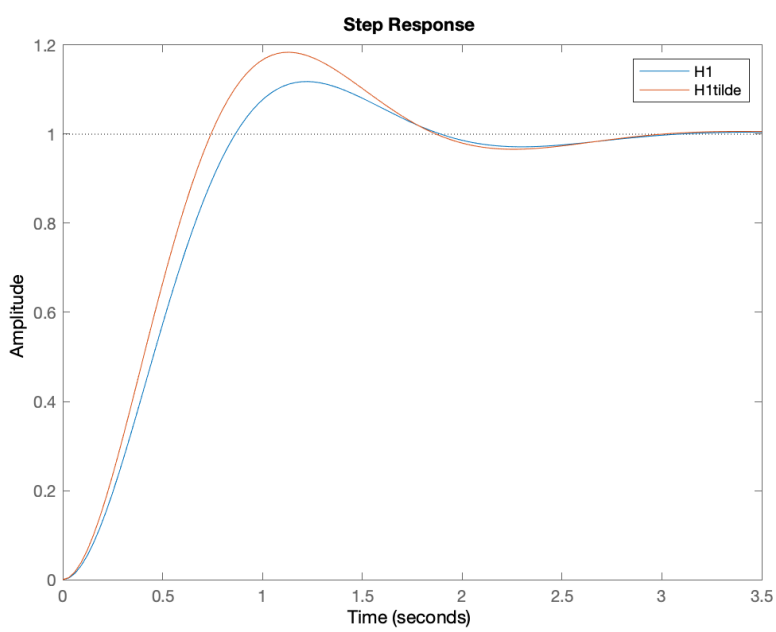


Figure 1: Step-response of the system represented by $H_1(s)$ and $\tilde{H}_1(s)$.

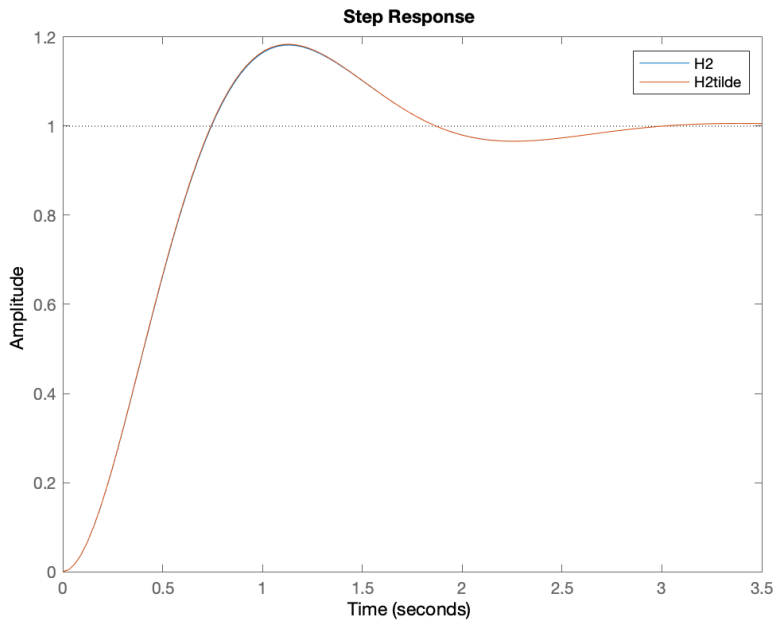


Figure 2: Step-response of the system represented by $H_2(s)$ and $\tilde{H}_2(s)$.

As can be seen, the pole cancellation works quite well on the $H_2(s)$ transfer function. So $\tilde{H}_2$ is a good approximation of $H_2(s)$. However, for $H_1(s)$, the pole cancellation is not a good approximation, since $\tilde{H}_1(s)$ incurs in a larger overshoot than $H_1(s)$ in the step response.

Problem 2

a. To solve it we will use the Laplace Transform, since:

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$$

Thus:

$$\begin{aligned}
(sI - A)^{-1} &= \begin{bmatrix} s+3 & -1 \\ 0 & s+4 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \frac{1}{s+3} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+4} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{s+3} & \frac{1}{s+3} - \frac{1}{s+4} \\ 0 & \frac{1}{s+4} \end{bmatrix}
\end{aligned}$$

Then we can take the inverse Laplace transform element-wise using the known LTs from the tables:

$$e^{At} = \begin{bmatrix} e^{-3t} & e^{-3t} - e^{-4t} \\ 0 & e^{-4t} \end{bmatrix}$$

b. Using convolution, the solution for $\boldsymbol{x}(t)$ is given by:

$$\begin{aligned}
\boldsymbol{x}(t) &= e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
&= \begin{bmatrix} e^{-3t} & e^{-3t} - e^{-4t} \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-3(t-\tau)} & e^{-3(t-\tau)} - e^{-4(t-\tau)} \\ 0 & e^{-4(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1d\tau \\
&= \begin{bmatrix} 3e^{-3t} - 2e^{-4t} \\ 2e^{-4t} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-3(t-\tau)} - e^{-4(t-\tau)} \\ e^{-4(t-\tau)} \end{bmatrix} d\tau \\
&= \begin{bmatrix} 3e^{-3t} - 2e^{-4t} \\ 2e^{-4t} \end{bmatrix} + \begin{bmatrix} \frac{1}{12}(1 - 4e^{-3t} + 3e^{-4t}) \\ \frac{1-e^{-4t}}{4} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{1 + 32e^{-3t} - 21e^{-4t}}{12} \\ \dfrac{1 + 7e^{-4t}}{4} \end{bmatrix}
\end{aligned}$$

and hence:

$$y(t) = x_1(t) + x_2(t) = \frac{4 + 32e^{-3t}}{12} = \frac{1 + 8e^{-3t}}{3}$$

c. We have:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -8e^{-3t} + 7e^{-4t} \\ -7e^{-4t} \end{bmatrix}$$

Then:

$$A\boldsymbol{x} = \begin{bmatrix} -3 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \dfrac{1 + 32e^{-3t} - 21e^{-4t}}{12} \\ \dfrac{1 + 7e^{-4t}}{4} \end{bmatrix} = \begin{bmatrix} -8e^{-3t} + 7e^{-4t} \\ -7e^{-4t} - 1 \end{bmatrix}$$

and since

$$Bu = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we have:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x} + Bu = \begin{bmatrix} -8e^{-3t} + 7e^{-4t} \\ -7e^{-4t} \end{bmatrix}$$

Problem 3

Taking the Laplace transform we have:

$$sX(s) - \boldsymbol{x}(0^-) = AX(s) + BU(s)$$

Rearranging terms we have:

$$(sI - A)X(s) = \boldsymbol{x}(0^-) + BU(s) \to X(s) = (sI - A)^{-1}(\boldsymbol{x}(0^-) + BU(s))$$

From problem 2 we have:

$$\begin{aligned}(sI - A)^{-1} &= \begin{bmatrix} s+3 & -1 \\ 0 & s+4 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{1}{s+3} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+4} \end{bmatrix}\end{aligned}$$

and hence (with $U(s) = 1/s$):

$$\begin{aligned}X(s) &= (sI - A)^{-1}(\boldsymbol{x}(0^-) + BU(s)) \\ &= \begin{bmatrix} \frac{1}{s+3} + \frac{2}{(s+3)(s+4)} \\ \frac{2}{s+4} \end{bmatrix} + \begin{bmatrix} \frac{1}{s(s+3)(s+4)} \\ \frac{1}{s(s+4)} \end{bmatrix}\end{aligned}$$

By doing partial fractions we obtain:

$$X(s) = \begin{bmatrix} \frac{1}{12s} + \frac{8}{3(s+3)} - \frac{7}{4(s+4)} \\ \frac{1}{4s} + \frac{7}{4(s+4)} \end{bmatrix}$$

By taking the inverse Laplace transform (for $t \geq 0$):

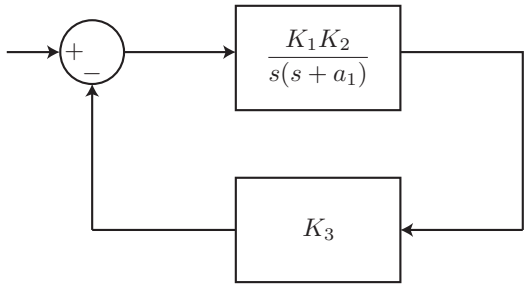
$$\boldsymbol{x}(t) = \begin{bmatrix} \frac{1}{12} + \frac{8}{3}e^{-3t} - \frac{7}{4}e^{-4t} \\ \frac{1}{4} + \frac{7}{4}e^{-4t} \end{bmatrix} = \begin{bmatrix} \frac{1+32e^{-3t}-21e^{-4t}}{12} \\ \frac{1+7e^{-4t}}{4} \end{bmatrix}$$

Finally:

$$y(t) = C\boldsymbol{x}(t) = \frac{1 + 8e^{-3t}}{3}$$

Problem 4

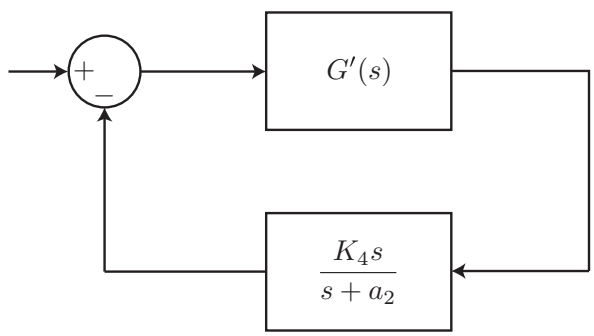
a. We work from inside to outside. Ignoring delays and grouping K_1 and K_2 we have:



We denote $G_1(s) = \frac{K_1K_2}{s(s+a_1)}$ and then the feedback transfer function is computed as:

$$\begin{aligned} G'(s) &= \frac{G_1(s)}{1+K_3G_1(s)} \\ &= \frac{\frac{K_1K_2}{s(s+a_1)}}{1+K_3\frac{K_1K_2}{s(s+a_1)}} \\ &= \frac{K_1K_2}{s(s+a_1)+K_1K_2K_3} \end{aligned}$$

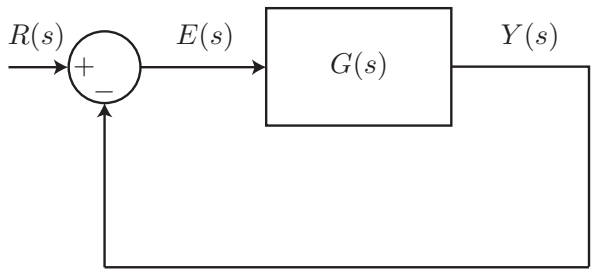
Then, we obtain:



Denoting $G_2(s) = \frac{K_4s}{s+a_2}$, the unity feedback system can be obtained as:

$$\begin{aligned} G(s) &= \frac{G'(s)}{1+G'(s)(G_2(s)-1)} \\ &= \frac{\frac{K_1K_2}{s(s+a_1)+K_1K_2K_3}}{1+\frac{K_1K_2}{s(s+a_1)+K_1K_2K_3}(G_2(s)-1)} \\ &= \frac{K_1K_2}{s(s+a_1)+K_1K_2K_3+K_1K_2(G_2(s)-1)} \\ &= \frac{K_1K_2}{s(s+a_1)+K_1K_2K_3+K_1K_2\left(\frac{K_4s}{s+a_2}-1\right)} \\ &= \frac{K_1K_2(s+a_2)}{[s(s+a_1)+K_1K_2K_3-K_1K_2](s+a_2)+K_1K_2K_4s} \end{aligned}$$

to finally obtain:



b. It is clear that $Y(s) = G(s)E(s)$ and $E(s) = R(s) - Y(s) \rightarrow Y(s) = R(s) - E(s)$. Thus:

$$R(s) - E(s) = G(s)E(s) \rightarrow R(s) = E(s)(1 + G(s)) \rightarrow \frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}$$

Thus,

$$\begin{aligned}
\frac{E(s)}{R(s)} &= \frac{1}{1 + G(s)} \\
&= \frac{1}{1 + \frac{K_1K_2(s+a_2)}{[s(s+a_1)+K_1K_2K_3-K_1K_2](s+a_2)+K_1K_2K_4s}} \\
&= \frac{[s(s+a_1) + K_1K_2K_3 - K_1K_2](s+a_2) + K_1K_2K_4s}{[s(s+a_1) + K_1K_2K_3 - K_1K_2](s+a_2) + K_1K_2K_4s + K_1K_2(s+a_2)} \\
&= \frac{[s(s+a_1) + K_1K_2K_3 - K_1K_2](s+a_2) + K_1K_2K_4s}{[s(s+a_1) + K_1K_2K_3](s+a_2) + K_1K_2K_4s} \\
&= \frac{s^3 + (a_1+a_2)s^2 + (a_1a_2 - K_1K_2 + K_1K_2K_3 + K_1K_2K_4)s - a_2(K_1K_2 - K_1K_2K_3)}{s^3 + (a_1+a_2)s^2 + (a_1a_2 + K_1K_2K_3 + K_1K_2K_4)s + a_2K_1K_2K_3}
\end{aligned}$$

Problem 5

a. We transform the system into a unity feedback form. Based on problem 4 we have:

$$\begin{aligned} G(s) &= \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)(H(s) - 1)} \\ &= \frac{1 \cdot \frac{10(s+10)}{s(s+2)}}{1 + 1 \cdot \frac{10(s+10)}{s(s+2)} \cdot (s + 4 - 1)} \\ &= \frac{10(s+10)}{s(s+2) + 10(s+10)(s+3)} \\ &= \frac{10s + 100}{11s^2 + 132s + 300} \end{aligned}$$

Note that both poles are on the left-hand side. To compute the type we first compute the following limits:

$$\begin{aligned} K_p &= \lim_{s \to 0} G(s) = \frac{100}{300} = \frac{1}{3} \\ K_v &= \lim_{s \to 0} sG(s) = 0 \\ K_a &= \lim_{s \to 0} s^2 G(s) = 0 \end{aligned}$$

Then, the appropriate input is a step, our system is type 0, and the appropriate static error constant is the *position constant*, K_p, associated with the step input.

b. Since the system is type 0, then a step is an input that will yield a constant error.

c. The error is simply computed as:

$$e_{\text{step}}(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 + 1/3} = \frac{3}{4}$$