Problem Set 4: Solutions

Problem 1
a. We have:

\[ C_1(s) = \frac{8(s + 2.5)}{s(s + 2)(s^2 + 3s + 10)} = \frac{8s + 20}{s(s + 2)(s^2 + 3s + 10)} = \frac{A}{s} + \frac{B}{s + 2} + \frac{Cs + D}{s^2 + 3s + 10} = \frac{A(s + 2)(s^2 + 3s + 10) + Bs(s^2 + 3s + 10) + (Cs + D)(s + 2)}{s(s + 2)(s^2 + 3s + 10)} \]

There are multiple ways to obtain the coefficients. The “official” way is to expand everything and compare coefficients of polynomials. An alternative, is to try “clever” values of \( s \) (usually the roots) to quickly obtain the coefficients solutions. Refer to Chapter 2 on the textbook for details on the partial fractions expansion.

Now comparing numerator, with \( s \to 0 \), it eliminates the \( B, C \) and \( D \) terms, and we have:

\[ A \cdot (0 + 2) \cdot (0 + 10) = 8 \cdot 0 + 20 \to 20A = 20 \to A = 1 \]

Similarly, for \( s \to -2 \) we have:

\[ B \cdot (-2) \cdot (2^2 + 3 \cdot (-2) + 10) = 8 \cdot (-2) + 20 \to -16B = 4 \to B = -\frac{1}{4} \]

For the last case, \( C \) and \( D \), there are no easy roots, so we will explore two techniques to obtain the coefficients.

- Similarly to the previous idea, you can just replace values that were not used. For example, for \( s = 1 \) we have (and replacing \( A \) and \( B \)):

\[ A \cdot (1 + 2) \cdot (1^2 + 3 + 10) + B(1^2 + 3 + 10) + (C + D) \cdot 1 \cdot (1 + 2) = 8 + 20 \]
\[ 42 + \frac{-1}{4} \cdot 14 + 3C + 3D = 28 \]
\[ 3C + 3D = \frac{-21}{2} \]

And for \( s = -1 \) we have:

\[ A \cdot (-1 + 2) \cdot (1^2 - 3 + 10) + B(-1)(1^2 - 3 + 10) + (-C + D) \cdot (-1) \cdot (-1 + 2) = -8 + 20 \]
\[ 8 + \frac{-1}{4} \cdot (-1) \cdot 8 + C - D = 12 \]
\[ C - D = 2 \]
Solving the $2 \times 2$ system we obtain:

\[ C = -\frac{3}{4}, \quad D = -\frac{11}{4} \]

- Now, the second alternative is the one described in the textbook based on matching coefficients via multiplying the lowest common denominator, $s(s + 2)(s^2 + 3s + 10)$:

\[ 8s + 20 = A(s + 2)(s^2 + 3s + 10) + B(s^2 + 3s + 10) + (Cs + D)s(s + 2) \]

Replacing $A$ and $B$ and expanding we have:

\[ 8s + 20 = \left( C + \frac{3}{4} \right) s^3 + \left( 2C + D + \frac{17}{4} \right) s^2 + \left( 2D + \frac{27}{2} \right) s + 20 \]

Balancing coefficients between the polynomials, we have three equations for two variables. From the $s^3$ coefficient we have:

\[ C + \frac{3}{4} = 0 \rightarrow C = -\frac{3}{4} \]

From the $s$ coefficient we have:

\[ 2D + \frac{27}{2} = 8 \rightarrow D = -\frac{11}{4} \]

The same results as the previous technique. We confirm that the coefficient associated with $s^2$ is zero:

\[ 2C + D + \frac{17}{4} = -2 \cdot \frac{3}{4} - \frac{11}{4} + \frac{17}{4} = 0 \]

as expected.

Finally, the partial fraction expansion is:

\[ C_1(s) = \frac{1}{s} - \frac{1}{4(s + 2)} + \frac{-3s - 11}{4(s^2 + 3s + 10)} \]

Observe that the residue (or coefficient) associated with the pole at $s = -2$ (related with the $(s+2)^{-1}$ term), which is closest to the zero at $-2.5$, is given by $-1/4$ and is NOT negligible in comparison to the other residues. This showcases that probably the pole-cancellation is not appropriate. We will verify this claim using the step response later.

b. Repeating the process from part a. we obtain:

\[ C_2(s) = \frac{1.005}{s} - \frac{0.00625}{s + 2} + \frac{-0.99875s - 3.002875}{s^2 + 3s + 10} \]

Observe now that the residue of the pole at $s = -2$, which is closest to the zero at $s = -2.01$, is $0.00625$ and seems to be negligible in comparison to the other residues (at least two order of magnitude below the other residues). So, this pole-cancellation seems to be appropriate.

The code used for MATLAB is as follows (note that $H_2(s)$ does not have a final value of 1, so we normalize by multiplying by $20/20.1$ in the numerator):
Listing 1: MATLAB code that plots the step responses.

```
num1 = [8 20];
den1 = [1 5 16 20];
num1tilde = 10;
den1tilde = [1 3 10];
num2 = (20/20.1)*[10 20.1];

H1 = tf(num1, den1);
H1tilde = tf(num1tilde, den1tilde);
H2 = tf(num2, den1);

fig1 = figure();
step(H1)
hold on
step(H1tilde)
legend({'H1', 'H1tilde'})
hold off

fig2 = figure();
step(H2)
hold on
step(H1tilde)
legend({'H2', 'H1tilde'})
```

with responses:

![Step Response](image)

Figure 1: Step-response of the system represented by $H_1(s)$ and $\tilde{H}_1(s)$. 
As can be seen, the pole cancellation works quite well on the $H_2(s)$ transfer function. So $\tilde{H}_2$ is a good approximation of $H_2(s)$. However, for $H_1(s)$, the pole cancellation is not a good approximation, since $\tilde{H}_1(s)$ incurs a larger overshoot than $H_1(s)$ in the step response.
Problem 2

a. To solve it we will use the Laplace Transform, since:

\[ e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} \]

Thus:

\[
(sI - A)^{-1} = \begin{bmatrix}
s + 3 & -1 \\
0 & s + 4
\end{bmatrix}^{-1}
= \begin{bmatrix}
\frac{1}{s+3} & \frac{1}{(s+3)(s+4)} \\
0 & \frac{1}{s+4}
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{s+3} & \frac{1}{s+3} - \frac{1}{s+4} \\
0 & \frac{1}{s+4}
\end{bmatrix}
\]

Then we can take the inverse Laplace transform element-wise using the known LTs from the tables:

\[ e^{At} = \begin{bmatrix}
e^{-3t} & e^{-3t} - e^{-4t} \\
0 & e^{-4t}
\end{bmatrix} \]

b. Using convolution, the solution for \( x(t) \) is given by:

\[
x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau
= \begin{bmatrix}
e^{-3t} & e^{-3t} - e^{-4t} \\
0 & e^{-4t}
\end{bmatrix} \begin{bmatrix}1 \\
\end{bmatrix} + \int_0^t \begin{bmatrix}e^{-3(t-\tau)} & e^{-3(t-\tau)} - e^{-4(t-\tau)} \\
0 & e^{-4(t-\tau)}
\end{bmatrix} \begin{bmatrix}0 \\
1
\end{bmatrix} d\tau
= \begin{bmatrix}3e^{-3t} - 2e^{-4t} \\
2e^{-4t}
\end{bmatrix} + \int_0^t \begin{bmatrix}e^{-3(t-\tau)} - e^{-4(t-\tau)} \\
e^{-4(t-\tau)}
\end{bmatrix} d\tau
= \begin{bmatrix}3e^{-3t} - 2e^{-4t} \\
2e^{-4t}
\end{bmatrix} + \frac{1}{12} \begin{bmatrix}1 - 4e^{-3t} + 3e^{-4t} \\
1 - e^{-4t}
\end{bmatrix}
\]

and hence:

\[ y(t) = x_1(t) + x_2(t) = \frac{4 + 32e^{-3t}}{12} = \frac{1 + 8e^{-3t}}{3} \]

c. We have:

\[ \dot{x}(t) = \begin{bmatrix}-8e^{-3t} + 7e^{-4t} \\
-7e^{-4t}
\end{bmatrix} \]
Then:
\[
Ax = \begin{bmatrix} -3 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 + 32e^{-3t} - 21e^{-4t} \\ \frac{12}{1 + 7e^{-4t}} \end{bmatrix} = \begin{bmatrix} -8e^{-3t} + 7e^{-4t} \\ -7e^{-4t} - 1 \end{bmatrix}
\]
and since
\[
Bu = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
we have:
\[
\dot{x}(t) = Ax + Bu = \begin{bmatrix} -8e^{-3t} + 7e^{-4t} \\ -7e^{-4t} \end{bmatrix}
\]
Problem 3
Taking the Laplace transform we have:

\[ sX(s) - x(0^-) = AX(s) + BU(s) \]

Rearranging terms we have:

\[(sI - A)X(s) = x(0^-) + BU(s) \rightarrow X(s) = (sI - A)^{-1}(x(0^-) + BU(s)) \]

From problem 2 we have:

\[
(sI - A)^{-1} = \begin{bmatrix}
  s + 3 & -1 \\
  0 & s + 4
\end{bmatrix}^{-1} = \begin{bmatrix}
  \frac{1}{s+3} & \frac{1}{(s+3)(s+4)} \\
  0 & \frac{1}{s+4}
\end{bmatrix}
\]

and hence (with \(U(s) = 1/s\)):

\[
X(s) = (sI - A)^{-1}(x(0^-) + BU(s)) = \begin{bmatrix}
  \frac{1}{s+3} + \frac{2}{(s+3)(s+4)} \\
  \frac{1}{(s+3)(s+4)}
\end{bmatrix} \begin{bmatrix}
  \frac{1}{s+3} + \frac{2}{(s+3)(s+4)} \\
  \frac{1}{s+4}
\end{bmatrix}
\]

By doing partial fractions we obtain:

\[
X(s) = \begin{bmatrix}
  \frac{1}{12s} + \frac{8}{3(s+3)} - \frac{7}{4(s+4)} \\
  \frac{1}{4s} + \frac{7}{4(s+4)}
\end{bmatrix}
\]

By taking the inverse Laplace transform (for \(t \geq 0\)):

\[
x(t) = \begin{bmatrix}
  \frac{1}{12} + \frac{8}{3}e^{-3t} - \frac{7}{4}e^{-4t} \\
  \frac{1}{4} + \frac{7}{4}e^{-4t}
\end{bmatrix} = \begin{bmatrix}
  \frac{1+32e^{-3t} - 21e^{-4t}}{1+7e^{-4t}} \\
  \frac{12e^{-4t}}{1+7e^{-4t}}
\end{bmatrix}
\]

Finally:

\[
y(t) = Cx(t) = \frac{1 + 8e^{-3t}}{3}
\]
Problem 4

a. We work from inside to outside. Ignoring delays and grouping $K_1$ and $K_2$ we have:

\[
\begin{align*}
K_1 K_2 & \frac{s}{s(a_1 + 1)} \\
+ & \\
K_3 & \\
\end{align*}
\]

We denote $G_1(s) = \frac{K_1 K_2}{s(a_1 + 1)}$ and then the feedback transfer function is computed as:

\[
G'(s) = \frac{G_1(s)}{1 + K_3 G_1(s)} = \frac{\frac{K_1 K_2}{s(a_1 + 1)}}{1 + K_3 \frac{K_1 K_2}{s(a_1 + 1)}} = \frac{K_1 K_2}{s(a_1 + 1) + K_1 K_2 K_3}
\]

Then, we obtain:

\[
\begin{align*}
+ & \\
G'(s) & \\
K_4 s & \frac{s}{s(a_2 + 1)}
\end{align*}
\]

Denoting $G_2(s) = \frac{K_4 s}{s(a_2 + 1)}$, the unity feedback system can be obtained as:

\[
G(s) = \frac{G'(s)}{1 + G'(s)(G_2(s) - 1)} = \frac{\frac{K_1 K_2}{s(a_1 + 1) + K_1 K_2 K_3}}{1 + \frac{K_1 K_2}{s(a_1 + 1) + K_1 K_2 K_3} (G_2(s) - 1)} = \frac{s(a_1 + 1) + K_1 K_2 K_3 + K_1 K_2 (G_2(s) - 1)}{K_1 K_2} = \frac{s(a_1 + 1) + K_1 K_2 K_3 + K_1 K_2 \left( \frac{K_4 s}{s(a_2 + 1)} - 1 \right)}{K_1 K_2} = \frac{K_1 K_2 (s + a_2)}{s(a_1 + 1) + K_1 K_2 K_3 - K_1 K_2 (s + a_2) + K_1 K_2 K_4 s}
\]
to finally obtain:

\[ R(s) \rightarrow E(s) \rightarrow G(s) \rightarrow Y(s) \]

b. It is clear that \( Y(s) = G(s)E(s) \) and \( E(s) = R(s) - Y(s) \) \( \rightarrow Y(s) = R(s) - E(s) \). Thus:

\[
R(s) - E(s) = G(s)E(s) \rightarrow R(s) = E(s)(1 + G(s)) \rightarrow \frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}
\]

Thus,

\[
\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} = \frac{1}{1 + \frac{K_1K_2(s+a_2)}{s+a_1} + K_1K_2K_3 + K_1K_2K_4 + K_1K_2K_4s}
\]

\[
= \frac{s(s + a_1) + K_1K_2K_3 - K_1K_2(s + a_2) + K_1K_2K_4s}{s(s + a_1) + K_1K_2K_3 - K_1K_2(s + a_2) + K_1K_2K_4s + K_1K_2(s + a_2)}
\]

\[
= \frac{s(s + a_1) + K_1K_2K_3 - K_1K_2(s + a_2) + K_1K_2K_4s}{s(s + a_1) + K_1K_2K_3(s + a_2) + K_1K_2K_4s}
\]

\[
= \frac{s^3 + (a_1 + a_2)s^2 + (a_1a_2 - K_1K_2 + K_1K_2K_3 + K_1K_2K_4)s - a_2(K_1K_2 - K_1K_2K_3)}{s^3 + (a_1 + a_2)s^2 + (a_1a_2 + K_1K_2K_3 + K_1K_2K_4)s + a_2K_1K_2K_3}
\]
Problem 5

a. We transform the system into a unity feedback form. Based on problem 4 we have:

\[ G(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)(H(s) - 1)} = \frac{1 \cdot \frac{10(s+10)}{s(s+2)}}{1 + 1 \cdot \frac{10(s+10)}{s(s+2)} \cdot (s + 4 - 1)} = \frac{10(s + 10)}{s(s + 2) + 10(s + 10)(s + 3)} = \frac{10s + 100}{11s^2 + 132s + 300} \]

Note that both poles are on the left-hand side. To compute the type we first compute the following limits:

\[ K_p = \lim_{s \to 0} G(s) = \frac{100}{300} = \frac{1}{3} \]
\[ K_v = \lim_{s \to 0} sG(s) = 0 \]
\[ K_a = \lim_{s \to 0} s^2G(s) = 0 \]

Then, the appropriate input is a step, our system is type 0, and the appropriate static error constant is the position constant, \( K_p \), associated with the step input.

b. Since the system is type 0, then a step is an input that will yield a constant error.

c. The error is simply computed as:

\[ e_{\text{step}}(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 + 1/3} = \frac{3}{4} \]