EE128 FALL 2020 MT I

Problem 1 (25 pts)
Each part is independent.

~~[7 pts]~~ a) Consider a single-input single-output system with input $u(t)$ and output $y(t)$ with transfer function for the system:

$$\frac{Y(s)}{U(s)} = \frac{s^4 + 2s^2 + 1}{s^4 + 6s^3 + 9s^2 + 1}$$

This transfer function can be implemented with the block diagram below. (Note that this diagram is a modified version of the Lec#3 handout.)

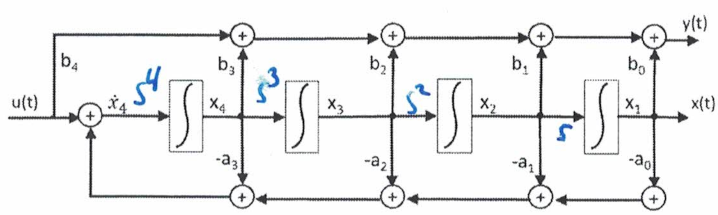
[4 pts] i) Write in the correct coefficients.

$a_0 =$ 1 $\quad a_1 =$ 0 $\quad a_2 =$ 9 $\quad a_3 =$ 6

$b_0 =$ 0 $\quad b_1 =$ 0 $\quad b_2 =$ -7 $\quad b_3 =$ -6 $\quad b_4 =$ 1

$s^4 X(s) = U(s) - (a_3 s^3 + a_2 s^2 + a_1 s + a_0) X(s)$

$$\frac{X(s)}{U(s)} = \frac{1}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$



$y = \dot{x}_4 + 2\ddot{x} + x = \dot{x}_4 + 2x_3 + x_1$

$\ddddot{x} = u - a_3 x_4 - a_2 x_3 - a_1 x_2 - a_0 x_1$

$y = u - a_3 x_4 - a_2 x_3 - a_1 x_2 - a_0 x_1$
$\quad + 2x_3 \quad + x_1$

$= u - 6x_4 - 7x_3$

~~[4~~ 2 pts] ii) For the output equation $y = C\mathbf{x} + Du = C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + b_4 u(t)$, find C.

$C = [$ 0 0 -7 -6 $]$

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9:53

[5 pts] b) A nonlinear system with output $x(t)$ and input $f(t)$ is described by the differential equation

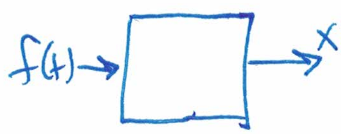
$$m\ddot{x} + b\dot{x} + \sin(x^3) = \log_e(1 + f(t))$$

The system input is a constant offset with small variations such that $f(t) = 1 + \delta f(t)$. The output $x(t) \approx x_o + \delta x(t)$. Find the transfer function relating output variation $\delta x(t)$ to input variation $\delta f(t)$.

$\log(2 + \delta f) \approx \log 2 + \frac{1}{2}\delta f$

$$\frac{\Delta X(s)}{\Delta F(s)} = \frac{1}{2(ms^2 + bs + \alpha)}$$

Where $\alpha = \cos(x_0^3)\, 3x_0^2$ at DC



$\sin x_0^3 = \log(2)$.

$\sin(x^3) \approx$ ~~f(x)~~ $g(x_0) + g'(x_0)\,\delta x = \sin(x_0^3) + \underbrace{\cos(x_0^3)\, 3x_0^2}_{\alpha} \cdot \delta x$

$\log(z) \approx \log(z_0) + \frac{1}{z}\Delta z$

$\frac{d}{dt}\sin(x^3) = \cos(x^3) \cdot 2x^2$

$m\ddot{\delta x} + b\dot{\delta x} + \cancel{\sin(x_0^3)} + \alpha\,\delta x = \frac{1}{2}\delta f + \cancel{\log(2)}$

$\Delta X(s)(ms^2 + bs + \alpha) = \frac{1}{2}\Delta F(s)$

Key.

[4 pts] c) Consider a system with a **step** input which has output transfer function:

$$Y(s) = \frac{40}{s(s+4)(s^2+2s+10)} = \frac{A}{s} + \frac{B}{s+4} + \frac{Cs+D}{s^2+2s+10}$$

Find the partial fraction expansion coefficients for $Y(s)$ (show work for full credit):

$A =$ 1

$B =$ $-5/9$

$C =$ $-4/9$

$D =$ $-28/9$

$A = sY(s)\big|_{s=0} = \frac{40}{4\cdot 10} = 1$, $B = (s+4)Y(s)\big|_{s=-4} = \frac{40}{(-4)(16-8+10)} = \frac{-10}{18} = -5/9$

$40 = 1\cdot(s+4)(s^2+2s+10) - \frac{5}{9}s(s^2+2s+10) + s\cdot(s+4)(Cs+D)$

$s^3 + 2s^2 + 10s$
$+4s^2 + 8s + 40$ $\quad -\frac{5}{9}s^3 - \frac{10}{9}s^2 - \frac{50}{9}s + Cs^3 + 4Cs^2 + Ds^2 + 4Ds$

s^3: $1 - \frac{5}{9} + C = 0 \Rightarrow C = -4/9$

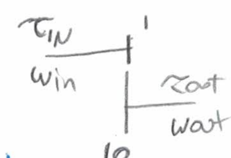
s^2: $6 - \frac{10}{9} - \frac{16}{9} + D = 0 \Rightarrow \frac{54}{9} - \frac{26}{9} + D = 0$, $D = -28/9$

[2 pts] d) Find $z(t)$, the inverse Laplace transform of $Z(s) = \frac{2}{s} + \frac{3}{s+4} + \frac{4s+5}{s^2+2s+10}$, showing work.

$z(t) =$ $\left(2 + 3e^{-4t} + 4e^{-t}\cos 3t + \frac{e^{-t}}{3}\sin 3t\right)u(t)$.

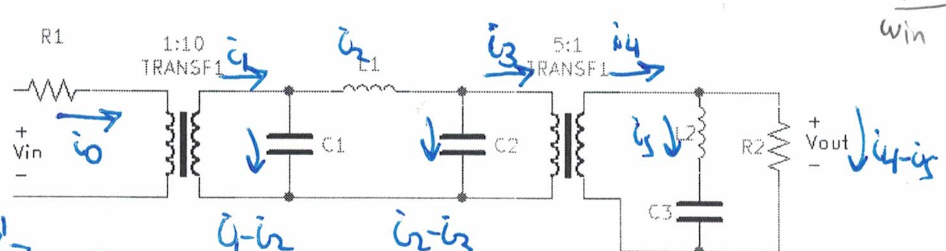
$$\frac{4s+5}{(s+1)^2+9} = \frac{4(s+1)}{(s+1)^2+9} + \frac{1}{(s+1)^2+9} \xrightarrow{\mathcal{L}^{-1}} 4e^{-t}\cos 3t + \frac{e^{-t}}{\omega_n}\sin 3t$$

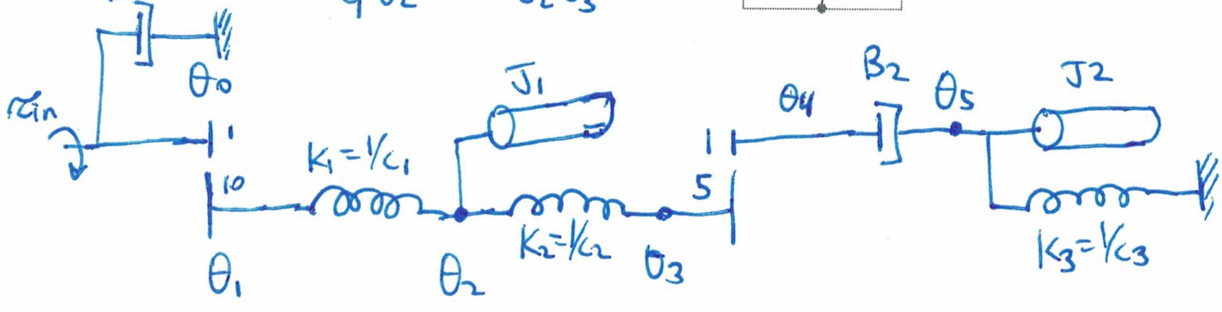
[8 pts] e) Draw the equivalent rotary mechanical circuit for this electrical system, with voltage corresponding to force and current to velocity. Input voltage is $V_{in}(t)$, output voltage is $V_{out}(t)$.



$\tau_{out} = 10\tau_{in}$

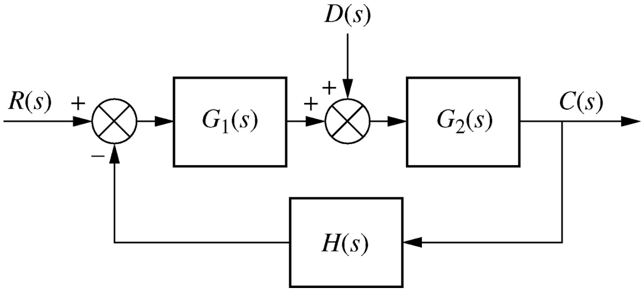
$\omega_{out} = \omega_{in}/10$





$\dot{\theta}_1 \sim i_1$

Problem 2 Steady State Error (16 pts)



For all parts, use:

$G_1(s) = k, \quad (k > 0)$

$G_2(s) = \frac{s+2}{(s+3)(s+4)}$

$H(s) = \frac{1}{s}$

[2 pts] a) Let $R(s) = 0$. Find the response to a general disturbance input $D(s)$ in terms of $G_1(s), G_2(s), H(s)$.

$\frac{C(s)}{D(s)} = \frac{G_2}{1+G_2G_1H}$

$$C = G_2\left(D + G_1(R - HC)\right)$$

$$C = G_2D + G_2G_1R - G_2G_1HC$$

$$(1+G_2G_1H)\,C = G_2D + G_2G_1R$$

[4 pts] b) For a disturbance input $d(t) = u(t)$, a unit step, (with $r(t) = 0$) find $\lim_{t\to\infty} c(t)$ (show work for full credit).

$$C = \frac{\frac{s+2}{(s+3)(s+4)}}{1 + \frac{k(s+2)}{(s+3)(s+4)}\cdot\frac{1}{s}} \cdot \frac{1}{s} = \frac{s+2}{s^3+7s^2+(12+k)s+2k}$$

$$\xrightarrow{\text{FVT}(k>0)} \lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) = \lim_{s\to 0} \frac{s^2+s}{\%} = \frac{0}{2k} = 0$$

$\rightarrow c(\infty) \rightarrow 0$

[4 pts] c) Let $e(t) = r(t) - c(t)$. Find $E(s)$ in terms of $G_1(s), G_2(s), H(s), R(s), D(s)$.

$$E(s) = \frac{1+G_2G_1H - G_2G_1}{1+G_2G_1} R(s) + \frac{-G_2}{1+G_2G_1H} D(s)$$

$$E = R - C = R - \frac{G_2D}{1+G_2G_1H} - \frac{G_2G_1R}{1+G_2G_1H} = \left(\frac{1+G_2G_1H - G_2G_1}{1+G_2G_1H}\right)R - \frac{G_2}{1+G_2G_1H}D$$

[6 pts] d) For $r(t) = tu(t)$, a unit ramp, and with $d(t) = u(t)$ a unit step, find $\lim_{t\to\infty} e(t)$ (the final value of the error) (show work for full credit).

From b) and linearity, the effect of D goes to 0.

$$e_\infty = \lim_{s\to 0} s \cdot \frac{1}{s^2} \cdot \frac{1+G_2G_1H - G_2G_1}{1+G_2G_1} = \lim_{s\to 0} \frac{1}{s}\left(\frac{1 + \frac{k(s+2)}{(s+3)(s+4)s} - \frac{k(s+2)}{(s+3)(s+4)}}{1+\frac{k(s+2)}{(s+3)(s+4)s}}\right)$$

$$= \lim_{s\to 0} \frac{1}{s}\cdot\left(\frac{s^3+(7-k)s^2+(10+k)s+2k}{s^3+7s^2+(12+k)s+2k}\right) \rightarrow \infty$$

explodes due to ramp

Problem 3. Root Locus Plotting (25 pts)

For the root locus $(1 + kG(s) = 0)$ with $k > 0$, and given open loop transfer function $G(s)$:

$$G(s) = \frac{(s+8)}{s(s+4)(s^2+6s+13)}$$

[1 pts] a) Determine the number of branches of the root locus = 4

[2 pts] b) Determine the locus of poles on the real axis $(-\infty, -8) \cup (-4, 0)$

[2 pts] c) Determine the angles for each asymptote: $60°, 180°, 240°$

[3 pts] d) determine the real axis intercept for the asymptotes (show work/formula) $\sigma = -2/3$

$$\frac{\sum p - \sum z}{\# p - \# z} = \frac{0-4-6+8}{4-1} = -\frac{2}{3}$$

[4 pts] e) Use the angle criteria for poles and zeros to find the $j\omega$ axis crossing point (show work).

Crossing at dj:

$$\operatorname{atan}\left(\frac{d}{8}\right) - \operatorname{atan}\left(\frac{d-2}{3}\right) - \operatorname{atan}\left(\frac{d}{4}\right) - \operatorname{atan}\left(\frac{d+2}{3}\right) - \frac{\pi}{2} = \pi$$

$\rightarrow$ via MATLAB $\rightarrow$ $d = \pm 2.85$

[4 pts] f) Estimate the value of k for the $j\omega$ axis crossing. (Show work for full credit).

$k = 29.35$

$$K = \frac{\prod l_{poles}}{\prod l_{zeros}} = \frac{2.85\sqrt{2.85^2+4^2}\sqrt{(2.85-2)^2+3^2}\sqrt{(2.85+2)^2+3^2}}{\sqrt{2.85^2+8^2}}$$

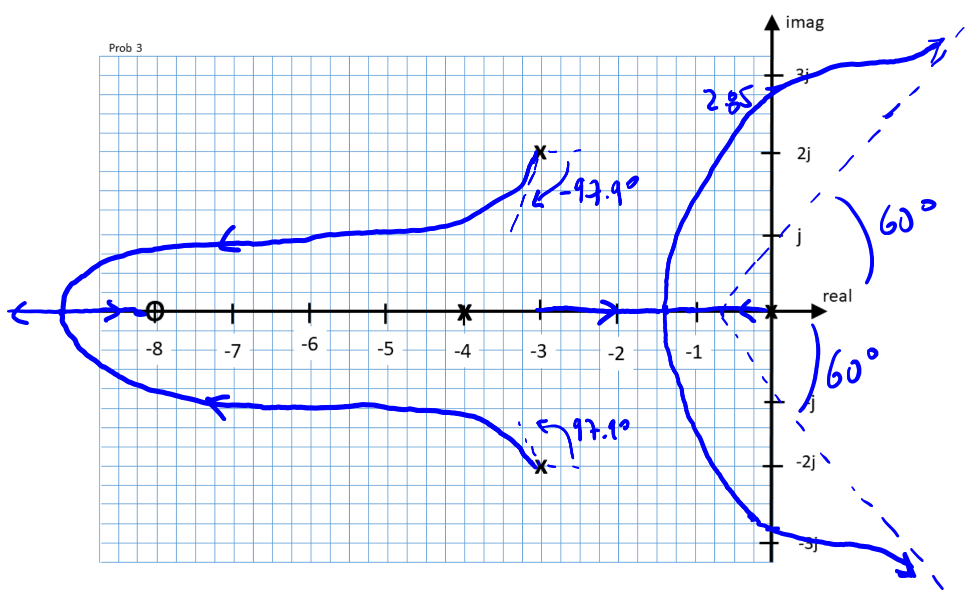
$$= 29.35$$

[4 pts] g) Calculate the angle of departure for the pole starting at $s = -3 + 2j$ (show work).

$$\theta_{dep} = \operatorname{atan}\left(\frac{2}{5}\right) - \operatorname{atan}\left(\frac{2}{1}\right) + \frac{\pi}{2} + \left(\frac{\pi}{2} + \operatorname{atan}\left(\frac{3}{2}\right)\right) + \pi$$

$$\approx -97.9°$$

[5 pts] h) Sketch a root locus below which is consistent with the information found above. Draw arrows on branches showing increasing gain. Draw asymptotes.



break in/away:

$-1.39, -9.85$

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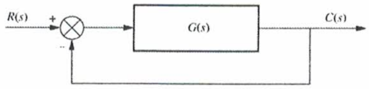
Problem 4. Root Locus and PD Compensation (~~18~~ pts)



Given open loop transfer function $G(s)$, where $G_1(s)$ is the open-loop plant:

$$G(s) = G_c(s)G_1(s) = G_c(s)\frac{s+10}{(s+2)^3}$$

and $G_c(s)$ is a PD compensation of the form $G_c(s) = k(s + z_c)$. The closed loop system, using unity gain feedback and the PD controller, should have a pair of poles at $p \approx -6 + 8.6j$ and $p^* \approx -6 - 8.6j$.

3 ~~[7 pts]~~ a. To obtain the closed loop pole at p, what is the angle contribution from the zero at $s = -10$? (show calculation)

$\angle(s + 10) =$ 65° $\quad \measuredangle 8.6j + 4 = \mathrm{atan}\left(\frac{8.6}{4}\right)$

3 ~~[7 pts]~~ b. To obtain the closed loop pole at p, what is the angle contribution from **each** pole at $s = -2$? (show calculation)

$\angle\frac{1}{s+2} =$ -115° $\quad = -\measuredangle(-4 + 8.6j)$

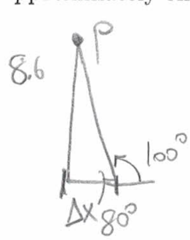
65° at $s = -10$

3 ~~[1]~~ pts c. What is the total angle contribution for all open-loop poles and the zero? -280°

[2 pts] d. What is the necessary angle contribution of the compensator zero z_c for the closed loop pole p to be on the root locus? $+100^\circ$

[3 pts] e. Find z_c to within ± 0.1 such that p is approximately on the root locus, within $\pm$ 2 degrees. (Show work.)

$z_c =$ $+4.5$



$\tan 80^\circ = \frac{8.6}{\Delta x}, \quad \Delta x = 1.5$

$6 - 1.5 = 4.5$

[2 pts] f. Add zero from PD compensation to pole-zero plot below and show angles on pole-zero plot below

[4 pts] g. Using lengths of vectors on the pole-zero plot, explain how to find k such that the closed loop pole p is on the root locus. Find k (show work).

$|KG(p)| = 1, \quad k = \frac{1}{|G|} = \frac{|P_1|^3}{|Z_1||Z_2|} = \frac{|P_1|^2}{|Z_2|} = \frac{9.5^2}{8.7}$

$k =$ 10.4

[2 pts] h. The step response for $\frac{G}{1+G}$ has faster rise time than the step response for the 2nd order approximation $\frac{pp*}{(s+p)(s+p*)}$. Briefly explain why.

t_r approx $= 0.17$

t_r PD $= .0992$

zero at -10 speeds ~~the~~ risetime up. (adds derivative).

$|P_1| = 9.5$

$|Z_1| = 9.5$

$|Z_2| = |1.5 + 8.6j| = 8.7$

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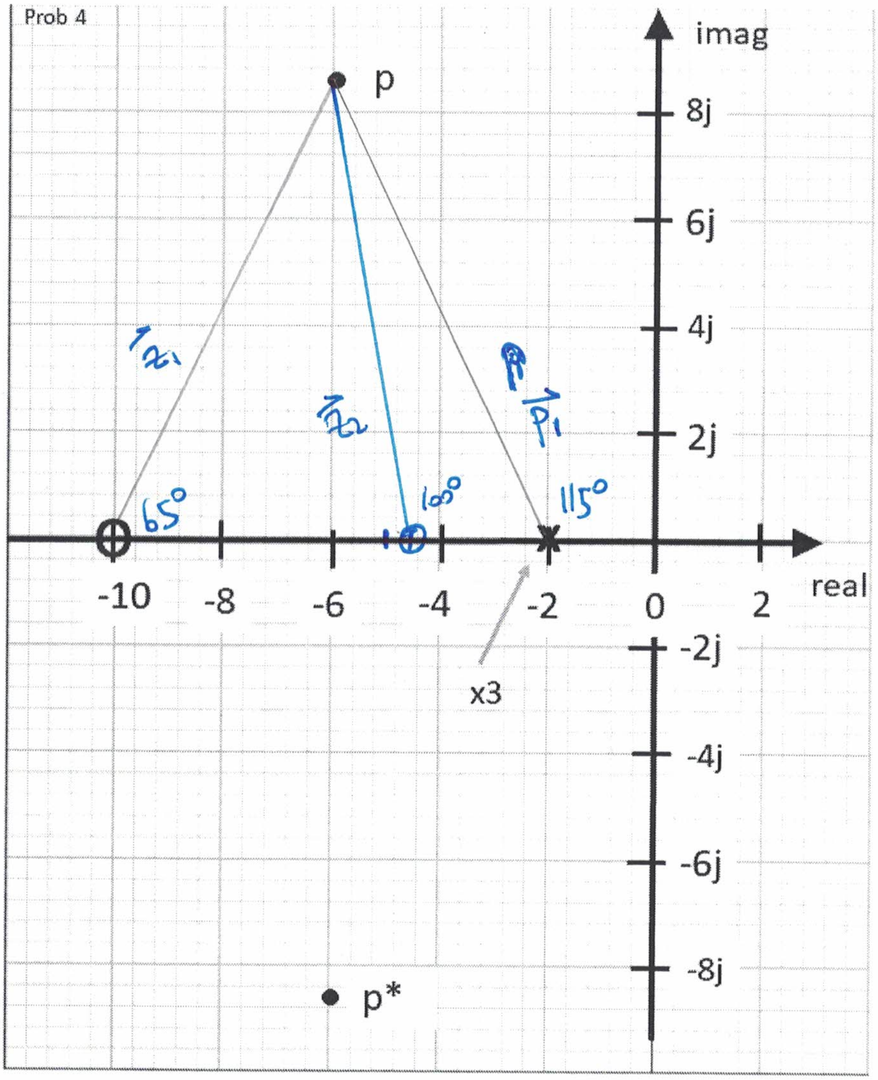
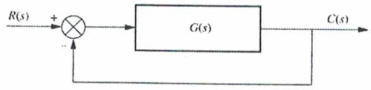
Pole-zero plot for PD compensation problem.



Problem 5. PI Compensation (12 pts)



Given open loop transfer function $G(s)$, where $G_2(s)$ is the open-loop plant:

$$G(s) = G_c(s)G_2(s) = G_c(s)\frac{s+4}{(s+5)^2(s+3)}$$

and $G_c(s)$ is a PI compensation of the form $G_c(s) = k\frac{s+z_c}{s}$. The closed loop system, using unity gain feedback and the PI controller, should have settling time $0.9 < T_s < 1.1$ sec and per cent over shoot $15\% < \%O.S. < 20\%$. (Note that $z_c > 0$ means that the zero is in the left half plane.)

Show/explain the procedures used to find z_c and k. Numerical parts can be from Matlab.

A_aw Work.

[8 pts] a. Find z_c and k to meet the settling time and per cent over shoot specification. (The step response should look approximately second order.)

$z_c =$ 2

$k =$ 70

$T_s \simeq \frac{4}{\zeta\omega_n}$, $\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}} \approx 0.5$

$\omega_n = \frac{4}{\zeta T_s} = \frac{4}{(0.5)(1)} = 8$

applies if 2nd order approx holds.

asymptotes $\pm\pi/2$

intercept $\frac{\Sigma p - \Sigma z}{4-2}$

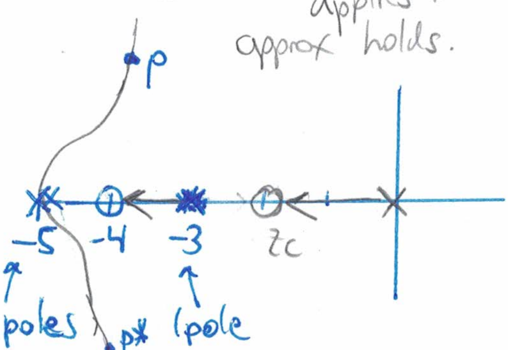
$\frac{-5-5-3-0-(-4-z_c)}{2}$

$= \frac{-9+z_c}{2}$, $z_c = 1$ is candidate. Check matlb.

Gain numerically from ~~K~~ Flows.

$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$= -4 \pm 6.9j$



[4 pts] b. Explain why the part of the step response due to the slow closed-loop pole is approximately cancelled by the zero you placed at z_c. Be quantitative, for example, using residues.

Matlab check for 2nd order

z_c	K	T_s	
1	115	1.7	← Slow pole not canceled
2	70	1.01, 18% OS.	

↑ good pole zero cancellation

4b. closed loop poles: for $\frac{G}{1+G}\cdot\frac{1}{s}$

$\frac{1}{s} - \frac{0.06}{s+1.89} - \frac{.0059}{s+3.98} - \frac{.467 \pm .19j}{s^2+7.1s\ 9.8}$

slow closed loop pole increases settling time

CLP at -3.98 cancelled by zero at -4

CLP at -1.89 only partially cancelled by zero at -2.

amplitude 1/8 2nd order component.

however, $-.06\, e^{-1.89 \cdot 1 sec} = .009$

which is less than 1% of final value.