Problem 1 (25 pts)
Each part is independent.

[7 pts] a) Consider a single-input single-output system with input \( u(t) \) and output \( y(t) \) with transfer function for the system:

\[
\frac{Y(s)}{U(s)} = \frac{s^4 + 2s^2 + 1}{s^4 + 6s^3 + 9s^2 + 1}
\]

This transfer function can be implemented with the block diagram below. (Note that this diagram is a modified version of the Lec#3 handout.)

[4 pts] i) Write in the correct coefficients.

\[
\begin{align*}
0 & \quad 0 & \quad 0 & \quad 1 & \quad 1 \\
0 & \quad 0 & \quad 0 & \quad -1 & \quad 1 \\
0 & \quad 0 & \quad 0 & \quad 0 & \quad 1 \\
-1 & \quad -1 & \quad -1 & \quad -1 & \quad -1
\end{align*}
\]

[4 pts] ii) For the output equation \( y = Cx + Du = C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + b_4 u(t), \) find \( C. \)

\[
C = \begin{bmatrix} 0 & 0 & -7 & 6 \end{bmatrix}
\]

[5 pts] b) A nonlinear system with output \( x(t) \) and input \( f(t) \) is described by the differential equation

\[
m\ddot{x} + b\dot{x} + \sin(x) = \log(1 + f(t))
\]

The system input is a constant offset with small variations such that \( f(t) = 1 + \delta f(t). \) The output \( x(t) \approx x_0 + \delta x(t). \) Find the transfer function relating output variation \( \delta x(t) \) to input variation \( \delta f(t). \)

\[
\frac{\Delta X(s)}{\Delta F(s)} = \frac{1}{2(m^2 + bs + d)}
\]

where \( d = \cos(\frac{x_0^3}{3})x_0^2 \)

\[
\log(2 + \delta f) \approx \log 2 + \frac{1}{2} \delta f
\]

\[
\Delta X(s)(ms^2 + bs + \alpha) = \frac{1}{2} \Delta F(s)
\]

\[
\sin x_0^3 = \log(2).
\]

\[
\frac{d}{dx} \sin(x^3) = \cos(x^3) \cdot 3x^2
\]

\[
\frac{\ddot{x}}{dx} + b\dot{x} + \sin(x_0^3) + \alpha \dot{x} = \frac{1}{2} \delta f + \log(1 + f(t))
\]
[4 pts] c) Consider a system with a step input which has output transfer function:

\[
Y(s) = \frac{40}{s(s+4)(s^2 + 2s + 10)} = \frac{A}{s} + \frac{B}{s+4} + \frac{Cs + D}{s^2 + 2s + 10}
\]

Find the partial fraction expansion coefficients for \( Y(s) \) (show work for full credit):

\[
A = \left. 5 Y(s) \right|_{s=0} = \frac{40}{40} = 1, \quad B = \left. (s+4)Y(s) \right|_{s=-4} = \frac{40}{16} = \frac{40}{16} = \frac{5}{16}
\]

\[
C = \left. \frac{s^2 + 2s + 10}{s^2 + 2s + 10} \right|_{s=-5/4} = \left. \frac{5}{9} \right|_{s=-5/4} = \frac{5}{9}
\]

\[
D = \left. \frac{5}{9} \right|_{s=-5/4} = \frac{5}{9}
\]

[2 pts] d) Find \( z(t) \), the inverse Laplace transform of \( Z(s) = \frac{2 + 3e^{-t} + 4e^{-t} \cos 3t + e^{-t} \sin 3t}{s^2 + 2s + 10} \), showing work.

\[
z(t) = \mathcal{L}^{-1}\left( \frac{2 + 3e^{-t} + 4e^{-t} \cos 3t + e^{-t} \sin 3t}{s^2 + 2s + 10} \right) = \frac{4(s+1)}{(s+1)^2 + 9} + \frac{1}{(s+1)^2 + 9} \to \frac{4e^{-t} \cos 3t + e^{-t} \sin 3t}{\omega_n}
\]

[8 pts] e) Draw the equivalent rotary mechanical circuit for this electrical system, with voltage corresponding to force and current to velocity. Input voltage is \( V_{in}(t) \), output voltage is \( V_{out}(t) \).
Problem 2 Steady State Error (16 pts)

For all parts, use:

\[ G_1(s) = k, \quad (k > 0) \]
\[ G_2(s) = \frac{s+2}{(s+3)(s+4)} \]
\[ H(s) = \frac{1}{s} \]

[2 pts] a) Let \( R(s) = 0 \). Find the response to a general disturbance input \( D(s) \) in terms of \( G_1(s), G_2(s), H(s) \).

\[
\frac{C(s)}{D(s)} = \frac{G_2}{1 + G_2 G_1 H} \quad \Rightarrow \quad C = G_2 D + G_2 G_1 R - G_2 G_1 H C
\]

\[
(1 + G_2 G_1 H) C = G_2 D + G_2 G_1 R
\]

[4 pts] b) For a disturbance input \( d(t) = u(t) \), a unit step, (with \( r(t) = 0 \)) find \( \lim_{t \to \infty} c(t) \) (show work for full credit).

\[
C = \frac{s+2}{(s+3)(s+4)} \cdot \frac{1}{s} = \frac{s+2}{s^2 + 7s^2 + (12+k)s + 2k}
\]

\[
\lim_{t \to \infty} c(t) = \lim_{s \to 0} C(s) = \lim_{s \to 0} \frac{s+2}{s^2 + 7s^2 + (12+k)s + 2k} = 0
\]

[4 pts] c) Let \( e(t) = r(t) - c(t) \). Find \( E(s) \) in terms of \( G_1(s), G_2(s), H(s), R(s), D(s) \).

\[
E(s) = \frac{1 + G_2 G_1 H - G_2 G_1}{1 + G_2 G_1 H} R(s) + \frac{-G_2}{1 + G_2 G_1 H} D(s)
\]

[6 pts] d) For \( r(t) = tu(t) \), a unit ramp, and with \( d(t) = u(t) \) a unit step, find \( \lim_{t \to \infty} e(t) \) (the final value of the error) (show work for full credit).

From b) and linearity, the effect of \( D \) goes to 0.

\[
e_{\infty} = \lim_{s \to 0} 1 \cdot \frac{1 + G_2 G_1 H - G_2 G_1}{1 + G_2 G_1 H} = \lim_{s \to 0} \frac{1 + \frac{k(s+2)}{(s+3)(s+4)} - \frac{k(s+2)}{(s+3)(s+4)}}{1 + \frac{k(s+2)}{(s+3)(s+4)}}
\]

\[
= \lim_{s \to 0} \frac{1}{s} \cdot \left( \frac{s^2 + (7-k)s^2 + ((10+k)s + 2k)}{s^2 + 7s^2 + (12+k)s + 2k} \right) \to \infty
\]

exploder due to ramp
Problem 3. Root Locus Plotting (25 pts)

For the root locus \((1 + kG(s) = 0)\) with \(k > 0\), and given open loop transfer function \(G(s)\):

\[
G(s) = \frac{(s + 8)}{s(s + 4)(s^2 + 6s + 13)}
\]

[1 pts] a) Determine the number of branches of the root locus = \(4\)

[2 pts] b) Determine the locus of poles on the real axis \((-\infty, -8) \cup (-4, 0)\)

[2 pts] c) Determine the angles for each asymptote: \(60^\circ, 180^\circ, 240^\circ\)

[3 pts] d) determine the real axis intercept for the asymptotes (show work/formula) \(\sigma = -2/3\)

\[
\sum p - \sum z = \frac{0 - 4 - 6 + 9}{4 - 1} = -\frac{2}{3}
\]

[4 pts] e) Use the angle criteria for poles and zeros to find the \(j\omega\) axis crossing point (show work).

\[
\tan^{-1}\left(\frac{\frac{1}{3}}{\frac{1}{4}}\right) - \tan^{-1}\left(\frac{\frac{1}{3}}{\frac{1}{4}}\right) - \tan^{-1}\left(\frac{\frac{1}{3}}{\frac{1}{4}}\right) - \tan^{-1}\left(\frac{\frac{1}{3}}{\frac{1}{4}}\right) = \pi
\]

\(\omega_{\text{MATLAB}} \rightarrow j = \pm 2.85\)

[4 pts] f) Estimate the value of \(k\) for the \(j\omega\) axis crossing. (Show work for full credit).

\[
k = 29.35
\]

\[
\frac{\pi \omega_{\text{MATLAB}}}{\pi \omega_{\text{peak}}} = \frac{2.85 \sqrt{2.85^2 + 8^2} \sqrt{(2.85 - 4)^2 + 3^2}}{2.85^2 + 8^2}
\]

\(= 29.35\)

[4 pts] g) Calculate the angle of departure for the pole starting at \(s = -3 + 2j\) (show work).

\[
\Theta_{\text{dep}} = \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) - \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{3}{2}}\right) + \pi + \left(\pi + \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{3}{2}}\right)\right) + \pi
\]

\(= -17.9^\circ\)

[5 pts] h) Sketch a root locus below which is consistent with the information found above. Draw arrows on branches showing increasing gain. Draw asymptotes.

![Root Locus Plot](image.png)
Problem 4. Root Locus and PD Compensation (18 pts)

Given open loop transfer function $G(s)$, where $G_1(s)$ is the open-loop plant:

$$G(s) = G_c(s)G_1(s) = G_c(s)\frac{s + 10}{(s + 2)^3}$$

and $G_c(s)$ is a PD compensation of the form $G_c(s) = k(s + z_c)$. The closed loop system, using unity gain feedback and the PD controller, should have a pair of poles at $p \approx -6 + 8.6j$ and $p^* \approx -6 - 8.6j$.

a. To obtain the closed loop pole at $p$, what is the angle contribution from the zero at $s = -10$? (show calculation)

$$\angle(s + 10) = \angle(10^0) = \angle(6j + 4) = \frac{180}{\pi} \angle(\theta)$$

b. To obtain the closed loop pole at $p$, what is the angle contribution from each pole at $s = -2$? (show calculation)

$$\angle\frac{1}{s+2} = -115^0 = -\angle(-4 + 6.6j)$$

c. What is the total angle contribution for all open-loop poles and the zero?

$$-280^0$$

d. What is the necessary angle contribution of the compensator zero $z_c$ for the closed loop pole $p$ to be on the root locus?

$$+100^0$$

e. Find $z_c$ to within ±0.1 such that $p$ is approximately on the root locus, within ± 2 degrees. (Show work.)

$$z_c = +4.5$$

f. Add zero from PD compensation to pole-zero plot below and show angles on pole-zero plot below

[4 pts] g. Using lengths of vectors on the pole-zero plot, explain how to find $k$ such that the closed loop pole $p$ is on the root locus. Find $k$ (show work).

$$k = \frac{10.4}{2}$$

[2 pts] h. The step response for $\frac{G}{1+G}$ has faster rise time than the step response for the 2nd order approximation $\frac{G}{(s+p)(s+p^*)}$. Briefly explain why.

$$tr \approx 0.17$$

$$tr_{PD} = 0.0992$$

$$kr = 9.5$$

$$k_r = 9.5$$

$$\frac{1}{2} = 1.5 + 6.6j$$
Pole-zero plot for PD compensation problem.
Problem 5. PI Compensation (12 pts)

Given open loop transfer function $G(s)$, where $G_2(s)$ is the open-loop plant:

$$G(s) = G_c(s)G_2(s) = G_c(s) \frac{s + 4}{(s + 5)^2(s + 3)}$$

and $G_c(s)$ is a PI compensation of the form $G_c(s) = k \frac{z_c}{s}$. The closed loop system, using unity gain feedback and the PI controller, should have settling time $0.9 < T_s < 1.1 \text{ sec}$ and per cent over shoot $15\% < \%O.S. < 20\%$. (Note that $z_c > 0$ means that the zero is in the left half plane.)

Show/explain the procedures used to find $z_c$ and $k$. Numerical parts can be from Matlab.

[8 pts] a. Find $z_c$ and $k$ to meet the settling time and per cent over shoot specification. (The step response should look approximately second order.)

$$z_c = \frac{2}{70}$$

$$k = \frac{2}{\text{Asymptotes } \pm \frac{\pi}{2}}$$

$$\text{Intercept } \frac{5 - z_c}{4 - 2}$$

$$-5 - 5 - 3 - 0 = (-4 - z_c)$$

$$z_c = \frac{9 + 2c}{2}$$

$c = 1$ is candidate. Check with $\text{Gain numerically from Hows.}$

$$s = -\frac{\text{wn}}{1 - \frac{\text{tc}}{s}}$$

$$-4 \pm 6.9 j$$

$$z_c = 1$$

[4 pts] b. Explain why the part of the step response due to the slow closed-loop pole is approximately cancelled by the zero you placed at $z_c$. Be quantitative, for example, using residues.

Matlab check for 2nd order

$$z_c \times k = \frac{1}{115} \times 1.7 \text{ Slow pole not cancelled}$$

2. 70 1.01, 186 as, good pole zero cancellation

4b. Closed loop poles for $\frac{G}{1 + G}$

$$\frac{1 - 0.06}{s + 1.89} - \frac{.0059}{s + 3.98} - \frac{.167 \pm 1.9j}{s^2 + 7.1s + 9.8}$$

Slow closed loop pole increases settling time

CLP at $-3.98$ cancelled by zero at $-4$.

CLP at $-1.89$ only partially cancelled by zero at $-2$.

Amplitude $\frac{1}{18}$ 2nd order component.

However, $-0.06 e^{-1.89t} sec = 0.09^9$ which is less than 1% of final value.