Key.

Problem 1 (22 pts)

A system described by a linear differential equation has input $u(t)$ and output $y(t)$:

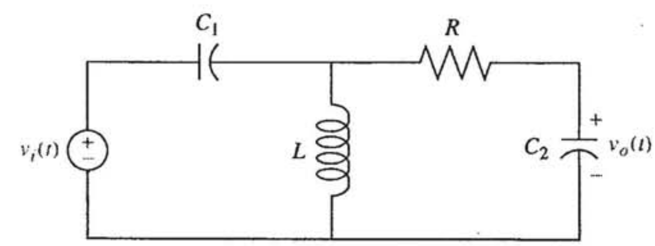
$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 20y = -u + \frac{du}{dt}$$

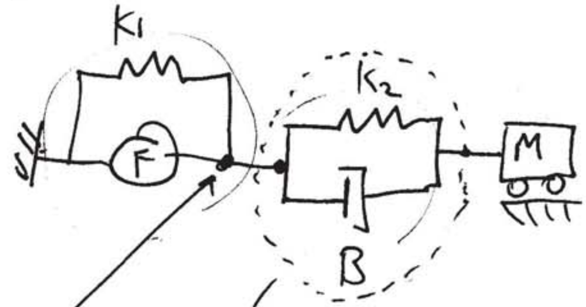
[5 pts] a) Assuming zero initial conditions:

$Y(s)(s^2+9s+20) = U(s)(s-1)$

Find the transfer function $\frac{Y(s)}{U(s)} = \frac{s-1}{s^2+9s+20}$

[8 pts] b) Draw the equivalent mechanical system for this circuit, with voltage corresponding to force and current to velocity. Let $C_1 = \frac{1}{K_1}, L = M, R = B, C_2 = \frac{1}{K_2}, v_i(t) = F_i(t)$.





Velocity here is sum of velocity M and velocity of (K_2, B)

note same force at each end.

[9 pts] c) A nonlinear system with input V_{in} and output V_{out} is described by the differential equation

$$\frac{V_{out}}{R} + C\frac{dV_{out}}{dt} = e^{V_{in}-V_{out}} - 1$$

For small V_{in} and V_{out}, find the transfer function for the linearized system:

$\frac{V_{out}(s)}{V_{in}(s)} =$ ________

V_{in} small, V_{out} small $\Rightarrow V_{in} - V_{out} \approx 0$.

$e^x \approx 1 + x + \frac{x^2}{2!} + \cdots \approx 1 + x$

$\frac{V_{out}}{R} + C\dot{V}_{out} \approx (1 + V_{in} - V_{out}) - 1 = V_{in} - V_{out}$

$V_{out}(s)(1 + 1/R + sC) = V_{in}(s)$

$\frac{V_{out}}{V_{in}} = \frac{1}{1 + 1/R + sC} = \frac{R}{R + 1 + sRC}$

$= \frac{1}{\frac{R+1}{R} + sC}$

Key

Problem 2 Steady State Error (22 pts)

For the system below, let $H(s) = \frac{s}{s+10}$, $G_1(s) = \frac{s+4}{s}$, and $G_2(s) = \frac{2}{s+1}$.

[5 pts] a) For $d(t) = 0$, and $r(t)$ a unit step, determine $C(s)$. $C(s) =$ ______

$$\frac{C}{R} = \frac{G_1 G_2}{1+G_1 G_2 H} = \frac{2(s+4)}{s(s+1) + \frac{2(s+4)s}{s+10}} = \frac{2(s+4)(s+10)}{s(s+10)(s+1) + 2(s+4)s}$$

$$C = \frac{C}{R}\cdot\frac{1}{s} = \frac{2}{s^2}\,\frac{(s+4)(s+10)}{(s+10)(s+1)+2(s+4)}$$

[6 pts] b) For $d(t) = 0$, and $r(t)$ a unit step, find $\lim_{t\to\infty} c(t) =$ ∞

$\lim_{t\to\infty} c(t) = s\,C(s)$ but $C(s)$ has $\frac{1}{s^2}$ term $\Rightarrow c(t) \to \infty$

[5 pts] c) For $d(t)$ a unit step and $r(t) = 0$, determine $C(s)$. $C(s) =$ ______

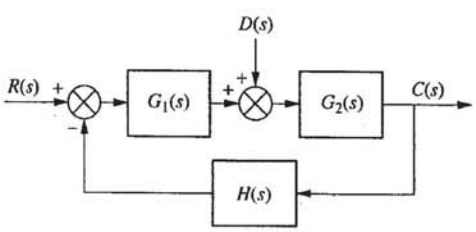
$$C = G_2(D + G_1(-HC))$$

$$C(1 + G_1 G_2 H) = G_2 D$$

$$\frac{C}{D} = \frac{G_2}{1+G_1 G_2 H}, \quad C(s) = \frac{D(s)}{s} = \frac{2}{s}\cdot\frac{2(s+10)}{(s+1)(s+10)+2(s+4)}$$

[6 pts] d) For $d(t)$ a unit step and $r(t) = 0$, find $\lim_{t\to\infty} c(t) =$ ______

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} \frac{2\cdot 10}{10+8} = \frac{20}{18} = 10/9$$



Problem 3. Routh-Hurwitz (15 pts)

Key.

Given open loop transfer function:

$$G(s) = \frac{k}{(s+3)^3}$$

and closed loop transfer function (assuming unity feedback)

$$T(s) = \frac{k}{s^3 + 9s^2 + 27s + 27 + k}$$

[10 pts] a. Using the Routh-Hurwitz table, find the range of k for which the closed loop system is stable.

$\underline{-27} < k < \underline{216}$

$$\begin{array}{llll} s^3 & 1 & 27 & 0 \\ s^2 & \cancel{9} & \cancel{27+k} & 0 \\ & 1 & 3+k/9 & \end{array}$$

$$s^1 = -\frac{\begin{vmatrix} 1 & 27 \\ 1 & 3+k/9 \end{vmatrix}}{1} = -(3+k/9-27) = 24-k/9 \Rightarrow 1 \quad 0 \quad 0$$

$$s^0 = -\frac{\begin{vmatrix} 1 & 3+k/9 \\ 1 & 0 \end{vmatrix}}{1} = 3+k/9$$

First column all positive.

#1 $24 - k/9 > 0$

$24 \cdot 9 > k$

$k < 216$

#2 $3 + \frac{k}{9} > 0$

$k > -27$

[5 pts] b. For the positive value of k found above, find the pair of closed loop poles on the imaginary axis.

$s = \pm j\omega_o = \pm \underline{j3\sqrt{3}}$

$k = 216 \Rightarrow s^1$ row of zeros $\Rightarrow s^2$ row for roots

$s^2: 9s^2 + 27 + 216 = 0$

$s^2 + 27 = 0$

$s = \pm j\sqrt{27} = \pm j3\sqrt{3}$

$1 + kG(j\omega) = 0$

$(j\omega + 3)^3 + 216 = 0$

$j\omega + 3 = \sqrt[3]{-216}$

$= 6\sqrt[3]{-1} = 6 \cdot e^{\pm j\pi/3}$

$= j3\sqrt{3}.$

Key.

Problem 4. Root Locus (17 pts)

Given open loop transfer function $G(s)$:

$$G(s) = \frac{(s+8)}{(s+1)(s+2)}$$

For the root locus $(1 + kG(s) = 0)$:

[2 pts] a) Determine the number of branches of the root locus = 2

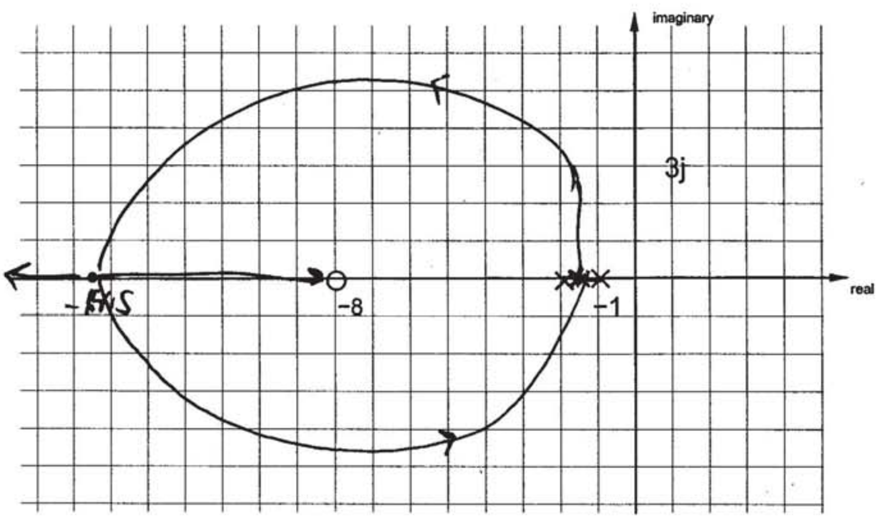
[2 pts] b) Determine the locus of poles on the real axis $s < -8, \; -2 < s < -1$

[2 pts] c) Determine the angles for each asymptote: -180°

asymptote $\dfrac{-(2l+1)\pi}{n-p} = -\pi$

[6 pts] d) The break away and break in points are at $s \approx -1.5$ and $s \approx -14.5$

[5 pts] e) Sketch the root locus below using the information found above.



Break away, Break in

$$\frac{1}{\sigma+1} + \frac{1}{\sigma+2} = \frac{1}{\sigma+8}$$

$$(\sigma+2)(\sigma+8) + (\sigma+1)(\sigma+8) = (\sigma+1)(\sigma+2)$$

$$\sigma^2 + 10\sigma + 16 + \sigma^2 + 9\sigma + 8 = \sigma^2 + 3\sigma + 2$$

$$\sigma^2 + 16\sigma + 22 = 0$$

$$\sigma = -8 \pm \frac{\sqrt{256-88}}{2}$$

$$= -8 \pm \sqrt{64-22}$$

$$= -8 \pm \sqrt{42}$$

$$36 < 42 < 49$$

$$\sqrt{42} \approx 6.5$$

Key.

Problem 5. Root Locus Compensation (24 pts)

Given open loop transfer function $G(s)$:

$$G(s) = G_1(s)\frac{1}{(s+3)^2(s+1)^2}$$

Where $G_1(s)$ is a PD control of the form $G_1(s) = k(s+\alpha)$.

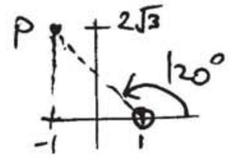
The closed loop system, using unity gain feedback and the PD controller, should have a pair of poles at $p = -1 \pm j2\sqrt{3}$.

[14 pts] a. Use the angle criteria to determine the zero location α for p to be on the root locus. Specify the angle contributions from each open loop pole. Mark the calculated zero on the pole-zero diagram below.

$\alpha = \underline{-1}$

$$\angle p = 2\angle \frac{1}{s+1} + 2\angle\frac{1}{s+3} + \angle(s+\alpha)$$

$$\angle(s+\alpha) = +120^\circ$$



$$\angle p = 2(-90) + 2(-60^\circ) + \angle(s+\alpha)$$

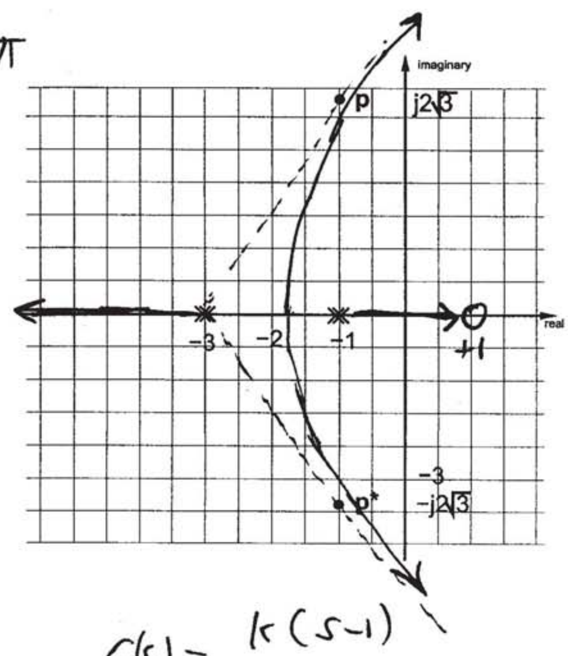
$$\underbrace{= -300^\circ + \angle(s+\alpha)}_{\text{must equal } -180^\circ}\ \left(\text{or } -(2l+1)\pi\right)\ldots$$

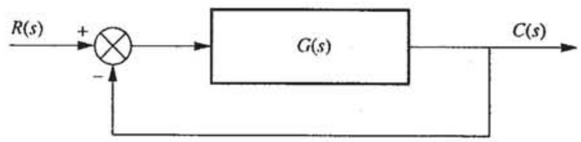
[10 pts] b. For the determined zero location, sketch the root locus, considering real-axis segments, real-axis intercept, and asymptotes.

4 branches $\theta = \frac{-(2l+1)\pi}{4-1} = \frac{\pi}{3}, -\frac{\pi}{3}, -\pi$

asymptote intercept:

$$\sigma = \frac{\Sigma p - \Sigma z}{3} = \frac{-3-3-1-1-(+1)}{3} = \frac{-9}{3} = -3$$





$$G(s) = \frac{k(s-1)}{(s+3)^2(s+1)^2}$$