Due at 1700, Fri. 4/21 in homework box under stairs, first floor Cory .
Note: up to 2 students may turn in a single writeup. Reading Nise 5, 12

1. (15 pts) Cayley-Hamilton (handout)
Given

$$A = \begin{bmatrix} -8 & 1 \\ -5 & -2 \end{bmatrix}. \tag{1}$$

By Cayley-Hamilton, $e^{At} = \alpha_0(t)I + \alpha_1(t)A$. Find $\alpha_0(t)I + \alpha_1(t)A$. Show that this e^{At} agrees with $e^{At} = \mathcal{L}^{-1}[sI - A]^{-1}$.

2. (35 pts) Output, State, and Observer Feedback (Separation principle handout)
Given the following system

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y = [10\ 0]\ \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} -10 \\ -5 \end{bmatrix}$$

[5pts] a) Design an output feedback controller $u = r - ky$ such that the system has a damping factor of $\zeta = \sqrt{2}/2$. Determine ω_n. Plot the step response using Matlab.
[5pts] b) Design a state feedback controller $u = r - [k_1\ \ k_2]\mathbf{x}$ such that the closed loop system has $\zeta = \sqrt{2}/2$ and ω_n which is twice the ω_n found in part a. Plot the step response using Matlab.
[6pts] c) Design a critically damped observer $\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + Bu + L(y - \hat{y})$ with both observer poles at $s = -25$.
[8pts] d) Write the state space equations for the controller with $u = r - K\hat{\mathbf{x}}$, such that ζ and ω_n are the same as in part b. (This should have 4 state variables, either $\mathbf{x}, \hat{\mathbf{x}}$ or $\mathbf{x}, \mathbf{e}$.) From the separation principle, what are the eigenvalues of the combined system with observer and feedback control? Plot the step response of the controller using the observer using Matlab.
[6pts] e) Use Matlab to plot the states $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ for $t > 0$ for the closed loop system of part d for a step input $r(t)$. (Suggestion, use `sys = ss(Ac,Bc,Cc,Dc)` and `[Y,T,X] = lsim(sys,r,t,x0)`, where `Ac, Bc, Cc, Dc` are the matrices for the system with observer).
[5pts] f) Compare the responses in part b) and d). What differences are there? (quantify).

3. (30 pts) Linear Quadratic Regulator (handout)
Consider two cars travelling in a straight line. The dynamics of car 1 are $\dot{x_2} = \ddot{x_1} = 2u_1$, and car 2 has a plant model $\dot{x_4} = \ddot{x_3} = 3u_2$ where u_1 and u_2 are the car's thrust due to engine and braking. (x_1 is a point 0.2 m behind car 1, and x_3 is the front bumper of car 2.) The outputs of the system are $y_1 = x_1$ and $y_2 = x_3 - x_1$. Note that if $y_2 > 0$ then car 2 has intruded on the safety zone of car 1.
Initial conditions are car 1 at -300 m, 30 m/sec, and car 2 at -310 m, 50 m/sec.
[4pts] a) Write the system equations in state space form.
[6pts] b) Use the LQR output method (Matlab function `lqry(sys,Q,R)`, with `Q=diag([1,1])` and `R=diag([1,1])` to find an optimal K for the state feedback control $\mathbf{u} = -K_b\mathbf{x}$. Plot $\mathbf{x}(t)$ and $\mathbf{u}(t)$ for the given initial condition (Matlab `initial`) and state feedback with gain K_b. How long does it take car 1 to get to within 1 m of the origin? What is car 1 velocity at 5 sec? Are there any problems with the system performance?
[12pts] c) Find new cost functions Q and R which maintains $y_2 < 0.2$ to prevent a collision, minimizes overshoot, and has both cars moving at less then 0.1 m/sec in 5 seconds. Plot $\mathbf{x}(t)$ and $\mathbf{u}(t)$ for the given initial condition (Matlab `initial`) and state feedback with new gain K_c.
[4pts] d) Find the solution to the Riccati equation P using Matlab function `care(A,B,Q,R)` and estimate the cost $J = (x^T P x)(0)$ for each of b) and c).
[4pts] e) Briefly compare the tradeoffs between control effort and time response between the two cases.

4. (20 pts) Discretization (Handout and Matlab)
Given the following continuous time (CT) system

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y = [1\ 0]\ \mathbf{x}$$

[10pts] a) Find the corresponding discrete time (DT) system $x[n+1] = Gx[n] + Hu[n]$, $y[n] = Cx[n]$ by hand calculation, with time step $T = \ln(1.5) sec$. Find the eigenvalues of G by hand.
[6pts] b) Find $x[1], x[2], x[3]$ with $u = 0$ and $x[0] = [0\ 1]^T$ by hand.
[4pts] c) Verify $x[k]$ for b) using Matlab `initial()` and turn in plot.