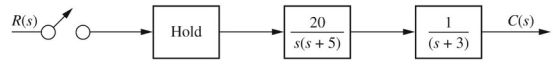
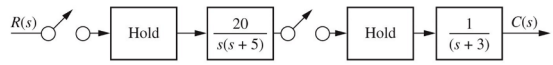
EE128 / ME134 Problem Set 12 Solutions Spring 2017

1. (10 pts) Discrete Time Conversion (Nise 13.4)
For each of the block diagrams below, find $\frac{C(z)}{R(z)}$ using $T = 0.5$ sec.



(a) ~~(b)~~



(b) ~~(a)~~

Solution:

a) $$\frac{1-e^{-Ts}}{s}\cdot\frac{20}{s(s+5)}\cdot\frac{1}{(s+3)} = \frac{4}{3}\frac{1}{s^2} - \frac{32}{45}\frac{1}{s} - \frac{2}{5}\frac{1}{s+5} + \frac{10}{9}\frac{1}{s+3}$$

$$\frac{C(z)}{R(z)} = \mathcal{Z}\left\{\frac{1-e^{-Ts}}{s}\cdot\frac{20}{s(s+5)}\cdot\frac{1}{(s+3)}\right\} = \frac{z-1}{z}\left(\frac{0.5z}{(z-1)^2}\cdot\frac{4}{3} - \frac{32}{45}\frac{z}{z-1} - \frac{2}{5}\frac{z}{z-e^{-2.5}} + \frac{10}{9}\frac{z}{z-e^{-1.5}}\right)$$

$$= \frac{1}{2}\frac{1}{z-1}\cdot\frac{4}{3} - \frac{32}{45} - \frac{2}{5}\frac{z-1}{z-0.0821} + \frac{10}{9}\frac{z-1}{z-0.2231}$$

$$= \frac{0.1706\,z^2 + 0.2815\cdot z + 0.0233}{(z-0.0821)(z-0.2231)(z-1)}$$

b) $$\mathcal{Z}\left\{\frac{1-e^{-Ts}}{s}\cdot\frac{20}{s(s+5)}\right\} = \frac{z-1}{z}\,\mathcal{Z}\left\{\frac{4}{s^2} + \frac{4}{5}\frac{1}{s+5} - \frac{4}{5}\frac{1}{s}\right\}$$

$$= \frac{z-1}{z}\left(\frac{2z}{(z-1)^2} + \frac{4}{5}\frac{z}{z-0.0821} - \frac{4}{5}\frac{z}{z-1}\right)$$

$$= \frac{1.266z + 0.5702}{(z-0.0821)(z-1)}$$

$$\mathcal{Z}\left\{\frac{1-e^{-Ts}}{s}\frac{1}{s+3}\right\} = \frac{z-1}{z}\,\mathcal{Z}\left\{\frac{1}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s+3}\right\} = \frac{z-1}{z}\left(\frac{1}{3}\frac{z}{z-1} - \frac{1}{3}\frac{z}{z-0.2231}\right)$$

$$= \frac{1}{3} - \frac{1}{3}\frac{z-1}{z-0.2231} = \frac{0.259}{z-0.2231}$$

$$\frac{C(z)}{R(z)} = \frac{1.266z + 0.5702}{(z-0.0821)(z-1)}\cdot\frac{0.259}{z-0.2231} = \frac{0.3279z + 0.1477}{(z-0.0821)(z-1)(z-0.2231)}$$

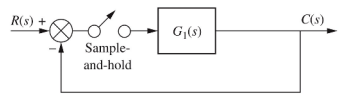
Their difference mainly comes from $\mathcal{Z}\{G_1(s)G_2(s)\} \neq G_1(z)\cdot G_2(z)$

System in (a) can be thought of as a linear system $G_1(s)\cdot G_2(s)$ discretized,

where $G_1(s) = \frac{1}{s+3}$, $G_2(s) = \frac{20}{s(s+5)}$

System in (b) can be thought of as two digital systems cascaded, thus $G_1(z)\cdot G_2(z)$

2. (15 pts) Stability of DT (Nise 13.6)
Find the range of gain K for which the system below with $G_1(s) = \frac{K}{s(s+4)}$ with $T = 0.25$ is stable.



Solution:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \quad \text{where} \quad G(s) = \frac{1-e^{-Ts}}{s} \cdot \frac{K}{s(s+4)}$$

$$G(z) = z\left\{ (1-e^{-Ts}) \frac{K}{s^2(s+4)} \right\}$$

$$= (1-z^{-1})\, z\left\{ \frac{K}{s^2(s+4)} \right\}$$

From Nise example 13.6 $\quad z\left\{ \frac{a}{s^2(s+a)} \right\} = \frac{Tz}{(z-1)^2} - \frac{(1-e^{-aT})z}{a(z-1)(z-e^{-aT})}$

$$\therefore G(z) = \frac{z-1}{z} \cdot \frac{K}{4} \cdot \left(\frac{Tz}{(z-1)^2} - \frac{(1-e^{-4T})z}{4(z-1)(z-e^{-4T})} \right)$$

$$= \frac{K}{4} \frac{T}{z-1} - \frac{K(1-e^{-4T})}{16} \cdot \frac{1}{z-e^{-4T}}$$

Put in $T = 0.25$

$$G(z) = \frac{K}{16} \frac{1}{z-1} - \frac{K}{16} \cdot \frac{1-e^{-1}}{z-e^{-1}}$$

$$= \frac{K}{16} \frac{z-e^{-1} - (1-e^{-1})(z-1)}{(z-1)(z-e^{-1})} = \frac{K}{16} \frac{e^{-1}z + (1-2e^{-1})}{(z-1)(z-e^{-1})}$$

Let $\hat{K} = K/16$.

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{\hat{K}(e^{-1}z + 1 - 2e^{-1})}{z^2 - (1+e^{-1})z + e^{-1} + e^{-1}\hat{K}z + (1-2e^{-1})\hat{K}}$$

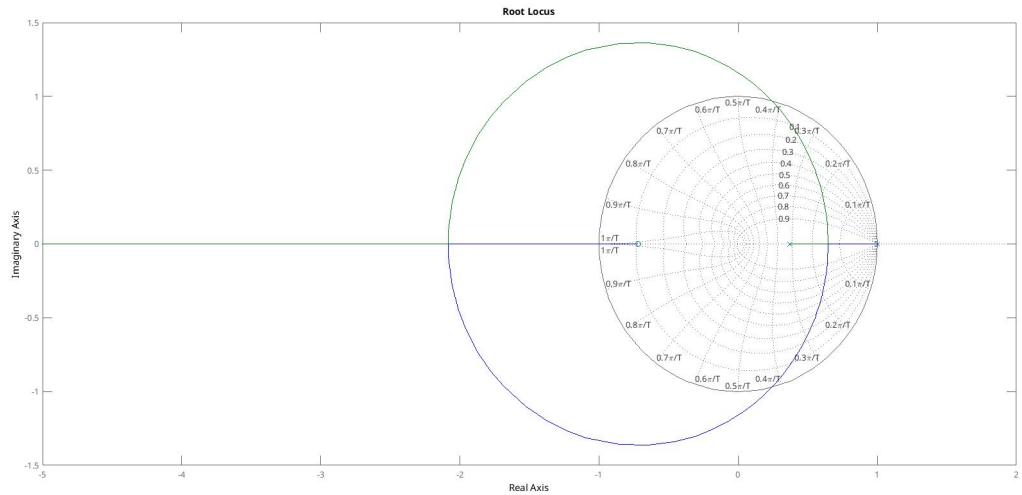
$$= \frac{\hat{K}(e^{-1}z + 1 - 2e^{-1})}{z^2 + (e^{-1}\hat{K} - 1 - e^{-1})z + (e^{-1} + (1-2e^{-1})\hat{K})}$$

$$= \frac{\hat{K}(0.3679z + 0.2642)}{z^2 + (0.3679\hat{K} - 1.3679)z + (0.3679 + 0.2642\hat{K})}$$

Poles are $\quad p_{1,2} = -\frac{1}{2}(0.3679\hat{K} - 1.3679) \pm \sqrt{\frac{1}{4}(0.3679\hat{K} - 1.3679)^2 - 0.3679 - 0.2642\hat{K}}$

$$= -0.1845\hat{K} + 0.6845 \pm \sqrt{0.0338\hat{K}^2 - 0.5158\hat{K} + 0.1}$$

Use Matlab to calculate root locus:



The stable range for k is $(0, 2.4)$

$\therefore \quad 0 < k < 38$

3. (20 pts) SS to TF (Nise 3.6, 13.3, 13.4, DT handout)
Given the following discrete time (DT) system, with sample period $T = 1$:

$$\mathbf{x}(k+1) = G\mathbf{x}(k) + Hu(k) = \begin{bmatrix} -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{2} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \tag{1}$$

[6pts] a. Find the transfer function $\frac{X(z)}{U(z)}$.
[2pts] b. Is the system BIBO stable?
[12pts] c. Find $x[n]$ for a unit step input and zero initial conditions using partial fraction expansion.

Solution:

a) $$\frac{X(z)}{U(z)} = (zI - G)^{-1} H = \begin{bmatrix} z+\frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & z \end{bmatrix}^{-1} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\left(z^2 + \frac{1}{4}z - \frac{3}{8}\right)} \begin{bmatrix} z & -\frac{3}{4} \\ -\frac{1}{2} & z+\frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-z - \frac{3}{4}}{z^2 + \frac{1}{4}z - \frac{3}{8}} \\ \dfrac{z + \frac{3}{4}}{z^2 + \frac{1}{4}z - \frac{3}{8}} \end{bmatrix}$$

b) Calculate poles $z^2 + \frac{1}{4}z - \frac{3}{8} = 0 \Rightarrow p_1 = -\frac{3}{4}$. $p_2 = \frac{1}{2}$ Both inside unit circle.

$\therefore$ BIBO stable.

c) $$\frac{X(z)}{U(z)} = \begin{bmatrix} \dfrac{-(z+\frac{3}{4})}{(z+\frac{3}{4})(z-\frac{1}{2})} \\ \dfrac{z+\frac{3}{4}}{(z+\frac{3}{4})(z-\frac{1}{2})} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{z-\frac{1}{2}} \\ \dfrac{1}{z-\frac{1}{2}} \end{bmatrix}$$

$$U(z) = \frac{z}{z-1}$$

$$x_1[n] = -x_2[n]$$

$$X_2(z) = \frac{z}{z-1} \cdot \frac{1}{z-\frac{1}{2}} = \frac{2z}{z-1} - \frac{2z}{z-\frac{1}{2}}$$

$$x_2[n] = 2 - 2\cdot\left(\frac{1}{2}\right)^n = 2 - \left(\frac{1}{2}\right)^{n-1}$$

$$x_1[n] = -x_2[n]$$

$$\therefore \quad x[n] = \begin{bmatrix} \left(\frac{1}{2}\right)^{n-1} - 2 \\ 2 - \left(\frac{1}{2}\right)^{n-1} \end{bmatrix}$$

4. (30 pts) Continuous vs Discrete Time Control (Handout and Matlab)

For each part, hand in relevant Matlab code as well as plots. Use `hold on` *to superimpose plots.*

Given the following continuous time (CT) system

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -25 & -35 & -11 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \quad y = [1\ 1\ 0]\ \mathbf{x}$$

[4pts] a) Find the corresponding discrete time (DT) system $x[n+1] = Gx[n] + Hu[n]$, $y[n] = Cx[n]$ which can be found using the Matlab function `c2d(ctsys,T,'zoh')`, with sampling period $T_s = 0.25$ sec. (Note `ctsys` can be found from `ss(A,B,C,D)`.) Compare eigenvalues for CT A and DT G; are both systems stable?

Solution:

a)

```
>> A = [0 1 0; 0 0 1; -25 -35 -11];
>> B = [0;0;1];
>> C=[10 1 0];
>> D=[0];
>> ctsys = ss(A,B,C,D);
>> dtsys = c2d(ctsys, 0.25, 'zoh')

  A =
          x1       x2       x3
   x1   0.9662   0.2002  0.01286
   x2  -0.3215    0.516  0.05876
   x3   -1.469   -2.378  -0.1304

  B =
          u1
   x1  0.001353
   x2   0.01286
   x3   0.05876

  C =
      x1  x2  x3
   y1  10   1   0

  D =
      u1
   y1  0

  Sample time: 0.25 seconds
  Discrete-time state-space model.
```

```
>> eig(A)
  -1.0000
  -5.0000
  -5.0000

>> eig(dtsys.A)
   0.7788
   0.2865
   0.2865
```

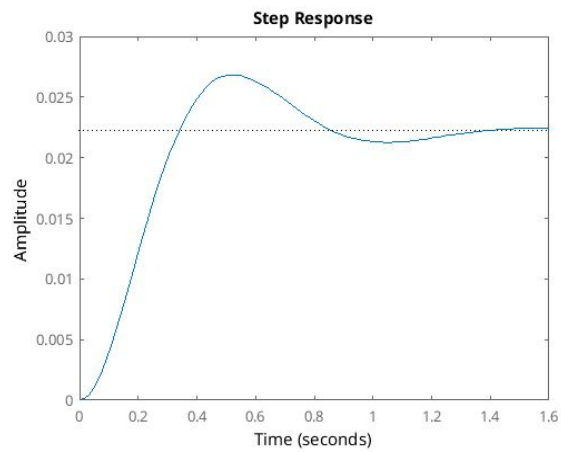
Both systems are stable.

[8pts] b) The continuous time system has state feedback such that $u(t) = r(t) - K\mathbf{x}(t)$. Find K such that closed loop poles are at $-10, -3 \pm 6j$. Plot the closed-loop step response using Matlab.

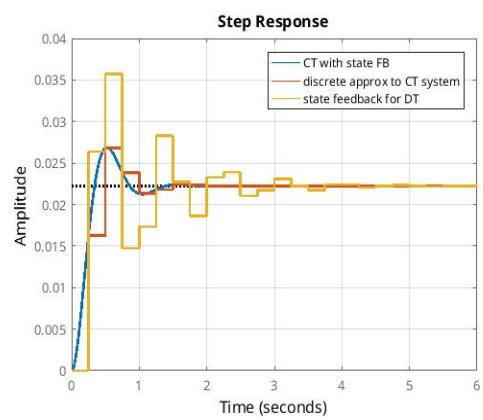
```
>> ps = [-10; -3+6i; -3-6i];
>> place(A, B, ps)
  425.0000   70.0000    5.0000
```

$\Rightarrow \quad K = [425, 70, 5]$

```
>> A = [0 1 0; 0 0 1; -25 -35 -11];
>> B = [0;0;1];
>> C=[10 1 0];
>> D=[0];
>> K=[425 70 5];
>> ctsys = ss(A,B,C,D);
>> ctfbsys = ss(A-B*K,B,C,D);
>> step(ctfbsys)
```



[6pts] c) Consider the CT system having state feedback applied in discrete time using a D/A converter such that $u[n] = u(nT) = r[n] - K\mathbf{x}[n]$ using the K found above. Thus the closed loop DT system has state equation $x[n+1] = (G - HK)x[n] + Hr[n]$, $y[n] = Cx[n]$. Plot the step response for the closed loop step response on the same axes as the CT step response of part b).



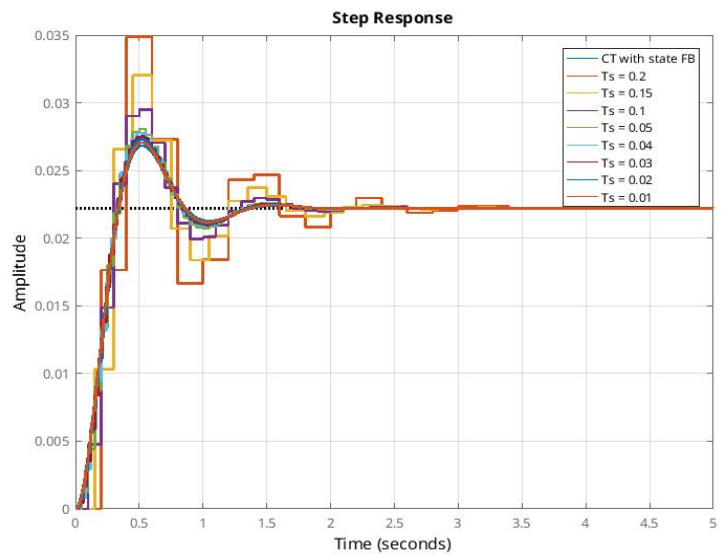

```
A = [0 1 0; 0 0 1; -25 -35 -11];
B = [0;0;1];
C=[10 1 0];
D=[0];
K=[425 70 5];
Ts = 0.25; % sampling time
ctsys = ss(A,B,C,D);
ctfbsys = ss(A-B*K,B,C,D);
dtsys = c2d(ctsys,Ts,'zoh');
dtwctfb = c2d(ctfbsys,Ts,'zoh');% discrete time but feedback in continuous time
G=dtsys.a;
H=dtsys.b;
dtfbsys = ss(G-H*K,H,C,D, Ts); % discrete feedback

figure(1)
step(clfctfbsys);
hold on;
step(dtwctfb);  % feedback in continuous time
step(dtfbsys);   % feedback in discrete time
set(findall(gcf,'Type','Text'),'FontSize',12)
set(findall(gcf,'Type','Line'),'LineWidth',2)
legend('CT with state FB','discrete approx to CT system', 'state feedback for DT')
```

[6pts] d) Explain why the state feedback for the DT system does not look like the DT version of $\dot{\mathbf{x}} = (A - BK)\mathbf{x} + Br$.

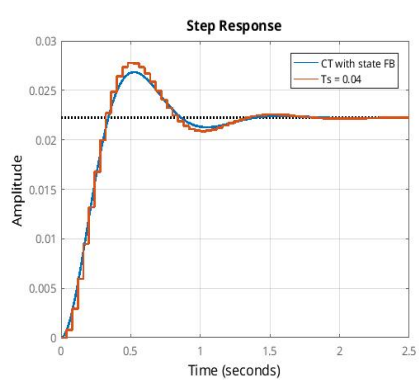
DT controller holds the control signal for a time step, so the input to the plant is not the same as the CT feedback. Because the control signal does not change as promptly as CT feedback, specs such as %OS and Ts are also worse.

[6pts] e) Use Matlab (iteratively if necessary) to find a sampling period T_s which gives a closed-loop step response for DT that is "reasonably close" (say within 5%) to the CT closed-loop step response. Determine closed-loop pole locations, and plot the DT step response on same axes as part c.

```
Gc = zpk([-10],[-5 -5 -1],1); % orig tf, zero at --10, poles at -5,-5, -1
A = [0 1 0; 0 0 1; -25 -35 -11];
B = [0;0;1];
C=[10 1 0];
D=[0];
K=[425 70 5];
Tss = [0.20:-0.05:0.05, 0.04:-0.01:0.01];
ctsys = ss(A,B,C,D);
ctfbsys = ss(A-B*K,B,C,D);
figure(1)
step(ctfbsys)
hold on;
legends = {'CT with state FB'};
for c = 1:length(Tss)
    Ts = Tss(c);
    dtsys = c2d(ctsys,Ts,'zoh');
    G=dtsys.a;
    H=dtsys.b;
    dtfbsys = ss(G-H*K,H,C,D, Ts);
    step(dtfbsys)
    legends{end+1} = sprintf('Ts = %g', Ts);
end
legend(legends{:});
set(findall(gcf,'Type','Text'),'FontSize',12)
set(findall(gcf,'Type','Line'),'LineWidth',2)
```



It seems that with any Ts below 0.05, the response looks "reasonably close".
Plot Ts=0.04 on top of part c).




```
>> eig(dtfbsys.A)
ans =
  0.8657 + 0.2214i
  0.8657 - 0.2214i
  0.6541 + 0.0000i
```

5. (20 pts) Transient performance using gain compensation (Nise 13.8,13.9)
Given a CT plant $G(s) = \frac{K}{(s+5)(s+1)}$.
[6pts] a. With sample period $T = 0.25$, find $G(z)$, the Z transform of $G(s)$.
[4pts] b. Sketch the root locus for $G(z)$ in unity gain feedback, and find the range of K for stability.
[4pts] c. With unity gain feedback, find the value of K for damping factor $\zeta = 0.6$, and note the K in root locus for CT and DT.
[6pts] d. Plot step response for the closed-loop CT and DT system in Matlab, and compare.

Solution:

a) $$G(z) = z\left\{\frac{K}{(s+5)(s+1)}\right\} = z\left\{\frac{K}{4}\frac{1}{s+1} - \frac{K}{4}\frac{1}{s+5}\right\}$$

$$= \frac{K}{4}\frac{z}{z-e^{-T}} - \frac{K}{4}\frac{z}{z-e^{-5T}}$$

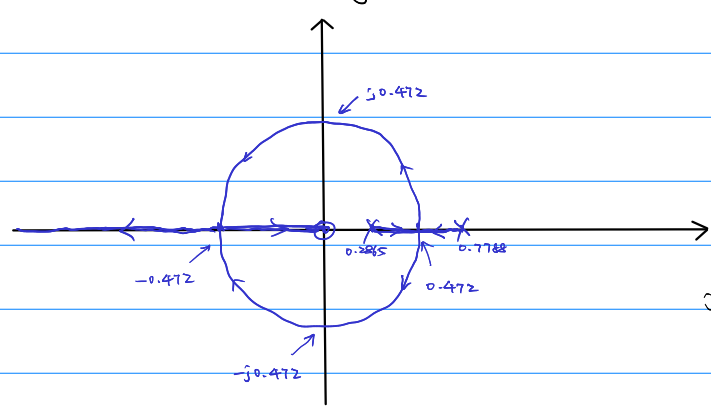
$$= \frac{K}{4}\frac{z}{z-0.7788} - \frac{K}{4}\frac{z}{z-0.2865}$$

$$= \frac{0.1231\,K\,z}{(z-0.7788)(z-0.2865)}$$

b) $$\frac{G(z)}{1+G(z)} = \frac{0.1231K\,z}{z^2+(0.1231K-1.0653)z+0.2231}$$

OL poles 0.7788, 0.2865 OL zero at 0

Real axis segments $(-\infty, 0]$, $[0.2865, 0.7788]$



Break-in/away points

$$\frac{1}{\sigma-0.7788} + \frac{1}{\sigma-0.2865} - \frac{1}{\sigma} = 0$$

$\Rightarrow \sigma_{1,2} = \pm 0.472$

Both inside unit circle

jw-axis crossing happens when

$0.1231K - 1.0653 = 0$

Crossings are at $z^2 + 0.2231 = 0$

$z_{1,2} = \pm j0.472$

Both inside unit circle.

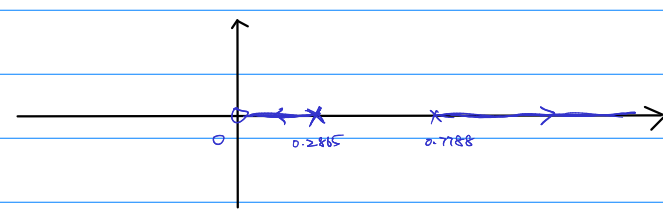
For CL system to be unstable, $K > K_1$

such that $z = -1$ is a root of $z^2 + (0.1231K_1 - 1.0653)z + 0.2231 = 0$

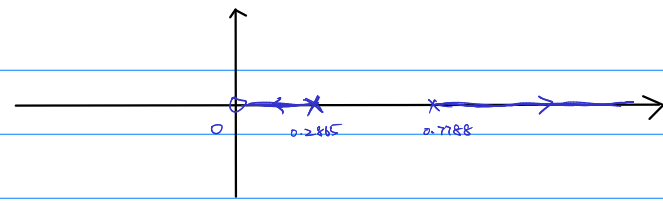
$\Rightarrow 1 - (0.1231K_1 - 1.0653) + 0.2231 = 0 \Rightarrow K_1 = 18.6$

$\therefore 0 \leq K < 18.6$

When $K < 0$, RL becomes



When $K < 0$, RL becomes



For CL system to be unstable, $K < K_2$

where $z = 1$ is a root of $z^2 + (0.1231K_2 - 1.0653)z + 0.2231 = 0$

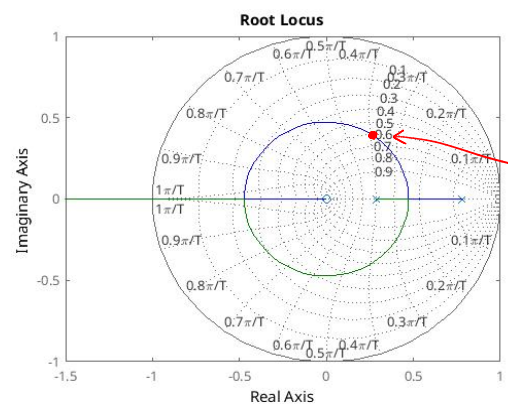
$\Rightarrow 1 + (0.1231 K_2 - 1.0653) + 0.2231 = 0 \Rightarrow K_2 \approx -1.28$

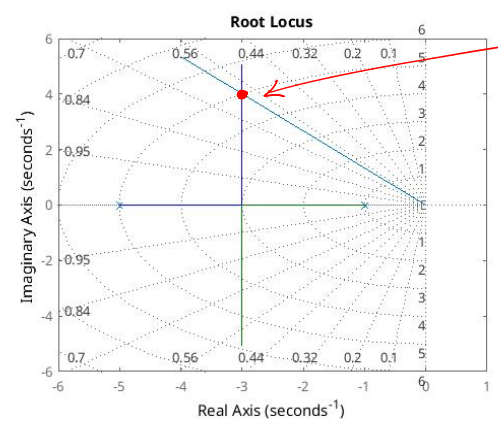
$\therefore -1.28 < K < 0$

From above, $-1.28 < K < 18.6$, system is stable.

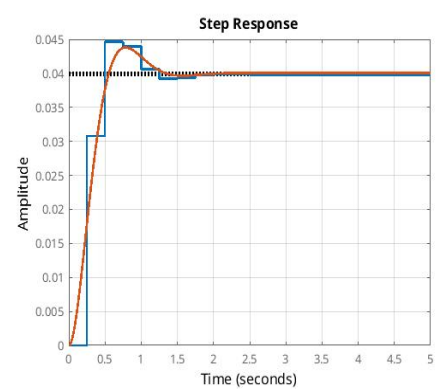
c)

```
Ts = 0.25
den = [1 6 5];
num = [1];
G = tf(num,den) % CTTF
Gimp = c2d(G,Ts,'impulse')
figure(1)
clf
rlocus(Gimp)
grid on
figure(2)
clf
k=0:0.01:30;
rlocus(G, k)
line([0, -4], [0, 4*tan(acos(0.6))]);
grid on
% plot step response
Hdt=feedback(Gimp,20);
Hct=feedback(G,20);
figure(3)
clf
hold on
step(Hdt)
step(Hct)
set(findall(gcf,'Type','Text'),'FontSize',12)
set(findall(gcf,'Type','Line'),'LineWidth',2)
grid on
```





$K = 20$ for $\xi = 0.6$



d) The step responses from CT and DT systems look reasonably close. The one from DT seems worse in both %OS and Ts.