EE128 / ME134 Problem Set 6 Solutions Spring 2017

1. (22 pts) Root locus (Nise 8.7)

Given the unity gain feedback system in Fig. 1, where

$$G(s) = \frac{K(s+15)(s+40)}{(s+30)(s^2-20s+200)}$$

`[11 pts]` a) Find and approximately hand sketch the root locus using RL rules 1-8.
`[4 pts]` b) Find the range of K which makes the system stable.
`[5 pts]` c) Using the second order approximation (assuming dominant 2nd order poles) find the value of K that gives $\zeta = 0.5$ and $T_s \approx 0.2$ for the system's dominant closed-loop poles.
`[2 pts]` e) Use MATLAB to plot the step response for c) and compare to approximation estimate.

Solution:

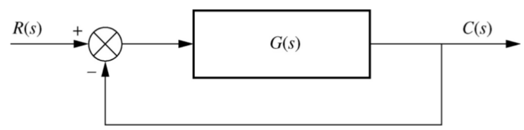


Fig. 1. Unity Gain Feedback.

a)

1) Number of RLs equal to number of CL poles

So there are 3 RLs

2) RLs are symmetric about real axis

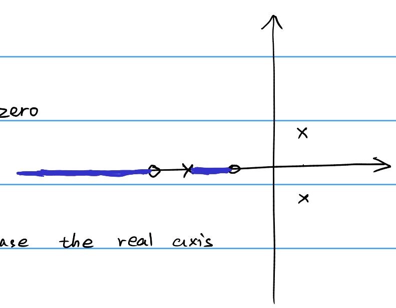
3) RLs exist to the left of odd numbers of finite poles and zeros

4) RLs begin at OL poles and end at OL zeros

OL poles $s = -30$, $s = 10 \pm 10j$

OL zeros $s = -15$, $s = -40$ and an infite zero

So far, we can draw the real axis segments of RLs as



5) Behaviour at infinity $\longrightarrow$ One RL appoaches straight line, in this case the real axis

6) Break-away / Break-in Locations

$$\sum_{i=1}^{m} \frac{1}{\sigma + z_i} = \sum_{i=1}^{n} \frac{1}{\sigma + p_i}$$ z_i and p_i are the negative of zeros and poles

$$\Rightarrow \frac{1}{\sigma+15} + \frac{1}{\sigma+40} = \frac{1}{\sigma+30} + \frac{1}{\sigma-(10+10j)} + \frac{1}{\sigma-(10-10j)}$$

$$\frac{1}{\sigma+15} + \frac{1}{\sigma+40} = \frac{1}{\sigma+30} + \frac{2\sigma-20}{\sigma^2-20\sigma+200}$$

$$(\sigma+30)(\sigma^2-20\sigma+200)(2\sigma+55) = (\sigma+15)(\sigma+40)(\sigma^2-20\sigma+200+2\sigma^2+40\sigma-600)$$

$$\sigma^4 + 110\sigma^3 + 2750\sigma^2 - 570000 = 0$$

roots: $\sigma_1 \approx -74.4388$

$\sigma_2 \approx 11.6882$

$\sigma_{3,4} \approx -23.6247 \pm j\,9.8490$

real axis break-in point is -74.4388

7) $j\omega$-axis crossing.

Use Routh table:

CL transfer function $\quad T(s) = \dfrac{K(s+15)(s+40)}{(s+30)(s^2-20s+200)+K(s+15)(s+40)}$

$$= \frac{K(s+15)(s+40)}{s^3+(K+10)s^2+(55K-400)s+(600K+6000)}$$

s^3	1	$55K-400$	0
s^2	$K+10$	$600K+6000$	0
s^1	$55K-400-600 = 55K-1000$	0	0
s^0	$600K+6000$	0	0

$K>0$, so both $K+10$ and $600K+6000$ are >0

For the system to be marginably stable, $55K-1000=0 \Rightarrow K=\frac{200}{11} \approx 18.1818$

when $K=\frac{200}{11}$, $\quad s^3+\frac{310}{11}s^2+600s+\frac{310}{11}\times 600$

roots $\quad s_1 = -\frac{310}{11} \approx -28.18$

$\quad s_{2,3} = \pm\sqrt{600} \approx \pm j24.49$

8) Angle of departure

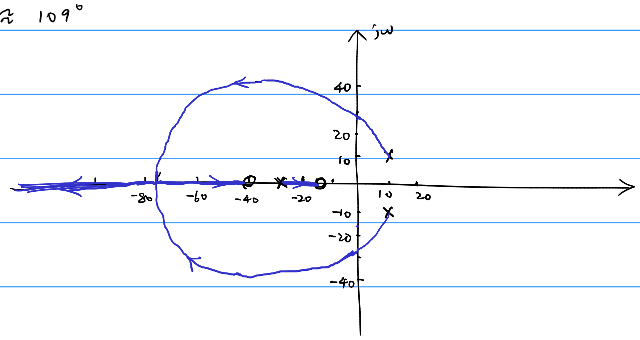
$$\sum_i^{m+n}\theta_i = (2k+1)\pi$$

$$-\theta_1-\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{1}{2.5}\right)+\tan^{-1}\left(\frac{1}{5}\right) = (2k+1)\pi$$

$$-\theta_1-1.2379 = (2k+1)\pi$$

$$\theta_1 \approx 109^\circ$$

From the above, the RLs can be drawn as



b) From a) -7), jω-axis crossing happens when $K=\frac{200}{11}\approx 18.18$.

When $K>18.18$, system is stable.

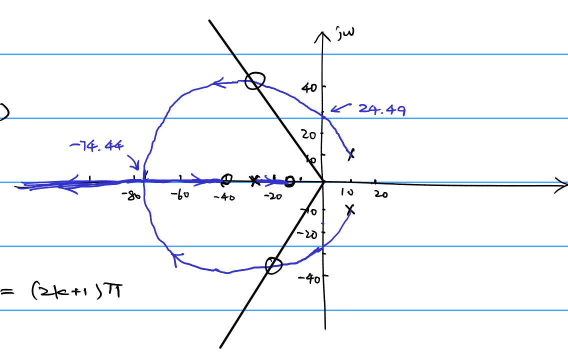
c) $\xi = 0.5 \Rightarrow$ dominant poles 60° from real axis

So we assume one of them is at $(-r, \sqrt{3}r)$

$$\sum_i^{m+n}\theta_i = (2k+1)\pi$$

$\Rightarrow \quad -\tan^{-1}\left(\frac{\sqrt{3}r-10}{-10-r}\right)-\tan^{-1}\left(\frac{\sqrt{3}r+10}{-10-r}\right)$

$-\tan^{-1}\left(\frac{\sqrt{3}r}{-r+30}\right)+\tan^{-1}\left(\frac{\sqrt{3}r}{-r+15}\right)+\tan^{-1}\left(\frac{\sqrt{3}r}{-r+40}\right) = (2k+1)\pi$



From the RL plot, r should be around $10 \sim 40$,

Do a line search, we can find $r \approx 20.07$

Dominant poles are at $-20.07 \pm j34.76$ $\omega_n \approx 2 \times 20.07 \approx 40.14$

Verify $T_s \approx \frac{4}{\xi \omega_n} \approx \frac{4}{0.5 \times 40.14} \approx 0.2\ s$

So the poles solved from ξ satisfy T_s as well

$$|G(p) \cdot H(p)| = \frac{1}{K}$$

$$\Rightarrow \frac{(34.76^2 + (20.07-15)^2)(34.76^2 + (20.07-40)^2)}{((+20.07-30)^2 + 34.76^2) \cdot ((20.07+10)^2 + (34.76-10)^2)((20.07+10)^2 + (34.76+10)^2)} = \frac{1}{K^2}$$

$$\Rightarrow K \approx 53.94 \approx 54$$

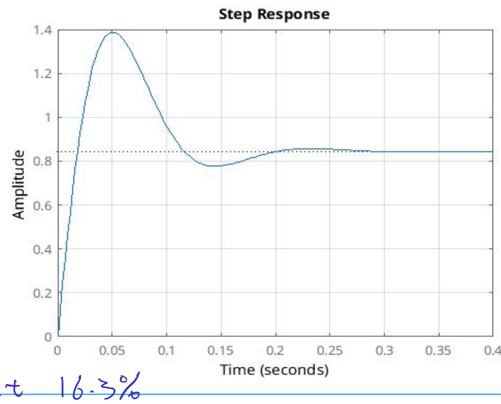
d) or e)

When $K = 54$ CL transfer function

$$T(s) = \frac{K(s+15)(s+40)}{s^3 + (K+10)s^2 + (55K-400)s + (600K+6000)}$$

$$= \frac{54(s+15)(s+40)}{s^3 + 64s^2 + 2570s + 38400}$$

Use MATLAB to plot step response:



Note: the %OS predicted from $\xi = 0.5$ is about 16.3%

The actual %OS is much larger, because pole doesn't quite cancel zero

2. (25 pts) Root locus (Nise 8.6, 8.9)

Consider the unity gain feedback system in Fig. 1 with $G(s) = \frac{k(s-10)(s-5)}{(s+20)(s+10)(s+2)}$. Here $-\infty < k < \infty$

[13 pts] a) Apply root locus rules: specify real axis segments, break-away and break-in locations on real axis, and angle of departure from complex poles.

[6 pts] b) Find the $j\omega$ crossing using Routh-Hurwitz.

[4 pts] c) Hand sketch the closed-loop root locus for positive and negative k.

[2 pts] d) Find the range of k for stability.

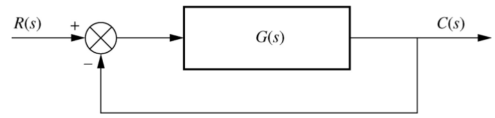


Fig. 1. Unity Gain Feedback.

Solution:

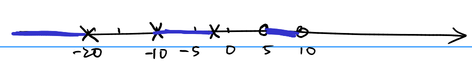
a) General information about RLs: for this system, RLs

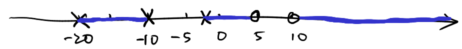
start from OL poles -20, -10, -2, and end at OL zeros

10, 5 and $-\infty$ $(k \to \infty)$ $+\infty$ $(k \to -\infty)$

For real axis segment, apply rule #4

When $k > 0$



When $K<0$ 

Break-away and break-in locations on real axis.

Apply rule #6

$$\sum_{i=1}^{m} \frac{1}{\sigma + z_i} = \sum_{i=1}^{n} \frac{1}{\sigma + p_i}$$

$$\frac{1}{\sigma - 5} + \frac{1}{\sigma - 10} = \frac{1}{\sigma + 20} + \frac{1}{\sigma + 10} + \frac{1}{\sigma + 5}$$

$$\Rightarrow \sigma^4 - 30\sigma^3 - 725\sigma^2 + 1510\sigma + 32500$$

$$\Rightarrow \sigma_1 \approx 45.01 \quad \sigma_2 \approx -15.12 \quad \sigma_3 \approx 6.96 \quad \sigma_2 \approx -6.86$$

Combined with real axis segments obtained from rule #4 :

When $K>0$, break-in at 6.96 . break-away at -6.86

when $K<0$, break-in at 45.01 break-away at -15.12

Angle of departure from complex poles

There are no complex poles

b) CL transfer function

$$T(s) = \frac{K(s-10)(s-5)}{K(s-10)(s-5) + (s+20)(s+10)(s+2)} = \frac{K(s-10)(s-5)}{s^3 + (K+32)s^2 + (-15K+260)s + (50K+400)}$$

Routh table

s^3	1	$-15K+260$	0
s^2	$K+32$	$50K+400$	0
s^1	$\frac{(K+32)(-15K+260)-(50K+400)}{K+32}$	0	0
s^0	$50K+400$	0	0

For first column to have no flipped signs.

$K>-32$, $K>-8$, $(K+32)(-15K+260)-(50K+400)>0$ $\Rightarrow$ $-33.68<K<15.68$

When K crosses -33.68 signs at first column change from $\begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}$ to $\begin{pmatrix} + \\ - \\ + \\ - \end{pmatrix}$

Two fewer flipped signs. $\Rightarrow$ this is a $j\omega$ crossing.

When K crosses 15.68 from $\begin{pmatrix} + \\ + \\ - \\ + \end{pmatrix}$ to $\begin{pmatrix} + \\ + \\ + \\ + \end{pmatrix}$

Two fewer flipped signs. $\Rightarrow$ this is a $j\omega$ crossing.

When K crosses -8, from $\begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}$ to $\begin{pmatrix} + \\ + \\ + \\ + \end{pmatrix}$, only one flipped sign.

It is a crossing, but on the real axis

When K crosses -32. ① when $K=-32+\varepsilon$. signs are $\begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}$

② when $K = -32-\varepsilon$ signs are $\begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}$

No change in the number of sign flips. not a crossing at all.

Then, find the crossings in each case.

$K = -33.68$ $s^3 - 1.68s^2 + (+15\times 33.68 + 260)s + (50\times(-33.68)+400)$

$\Rightarrow$ $s_1 \approx 1.68$ $s_{2,3} \approx \pm j27.66$

Crossings are $\pm j27.66$

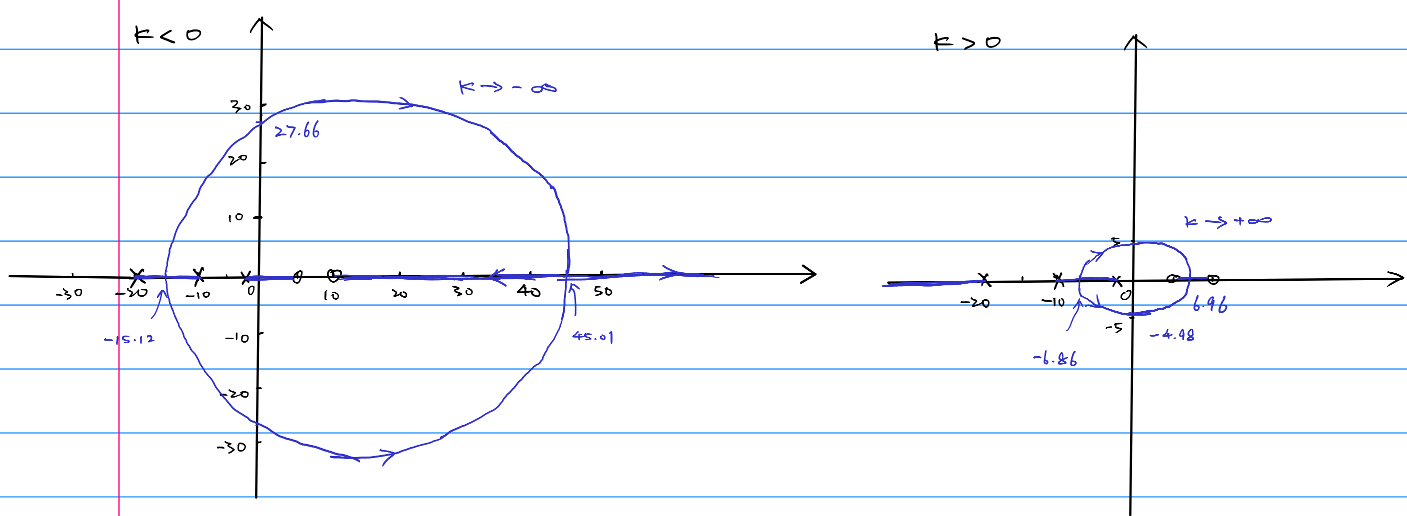
$K = 15.68$ $s^3 + 47.68s^2 + (-15\times 15.68 + 260)s + (50\times 15.68 + 400)$

$\Rightarrow$ $s_1 \approx -47.68$ $s_{2,3} \approx \pm j4.98$

Crossings are $\pm j4.98$

$K=-8$, it is a $j\omega$ axis crossing along the real axis, so it is at 0

c) From the discussions in a) and b)



d) From the discussion in b) and RLs in c) the range is $-8 < K < 15.68$

Or it should be better written as $-8 < K < 0$, $0 < K < 15.68$

Because the system is not connected at $K=0$

3. (26 pts) PI compensation (Nise 9.2)
Consider open loop plant

$$G(s) = \frac{2000K}{(s+10)^2(s+20)}$$

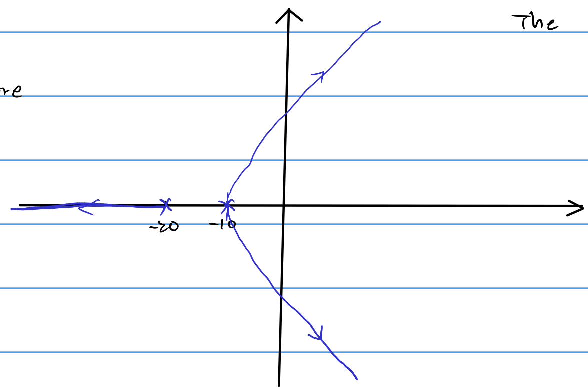
and unity feedback.
`[5 pts]` a) find K such that overshoot is 20%.
`[11 pts]` b) Design a PI controller with the same 20% overshoot such that steady state error is 0, with $T_s \leq 1$ sec. (Hint: PS4-1)
`[6 pts]` c) Hand sketch the root locus for the original system and the system with a **PI** compensator, and verify with Matlab.
`[2 pts]` d) Use Matlab to compare the step response for the closed-loop compensated and uncompensated systems, transient and steady state response.
`[2 pts]` e) Find the steady state error for a step for both systems.

Solution:

a) First, hand sketch root locus. Three OL poles, -10, -10, -20, real segment is $(-\infty, -20]$

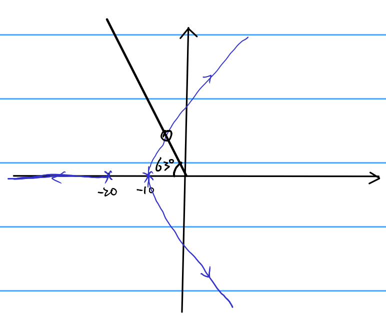
The other two RLs approach $\pm 60°$ lines

Therefore



With %OS ≈ 20%, we want $\xi = \sqrt{\dfrac{(\ln 0.2)^2}{\pi^2 + (\ln 0.2)^2}} \approx 0.456$

So dominant poles are $\cos^{-1}(0.456) \approx 63°$ from real axis



$\tan(63°) \approx 2.0$

So we assume one CL pole is at $(-r, 2r)$

$$\sum_i^{m+n} \theta_i = (2k+1)\pi$$

$$\Rightarrow -2\tan^{-1}\left(\frac{2r}{10-r}\right) - \tan^{-1}\left(\frac{2r}{20-r}\right) = (2k+1)\pi$$

$\Rightarrow r \approx 5.8$ dominant poles at $-5.8 \pm j11.6$

Find corresponding K, $|G(p)H(p)| = \frac{1}{K}$

$$\frac{2000^2}{((10-5.8)^2+11.6^2)^2 + ((20-5.8)^2+11.6^2)} = \frac{1}{K^2} \Rightarrow K \approx 1.4$$

b) Ts without controller $\rightarrow$ $T_s \approx \dfrac{4}{\xi \cdot \omega_n} \approx \dfrac{4}{0.456 \cdot \sqrt{5.8^2+11.6^2}} \approx 0.68 < 1s$

So we'd like to keep these poles almost unchanged, but use PI controller to get rid of the SS error.

A PI controller introduces an OL pole at 0, we want to keep the dominant poles almost unchanged. Therefore, we introduce a new zero somewhere before -10, i.e. within $(-10, 0)$

So we design PI controller as $G_c(s) = K\dfrac{s+5}{s}$

With the same $k = 1.4$, Let's verify that it meets the specs using second order approximation

$$G(s) \cdot G_c(s) = \frac{2000k\,(s+5)}{(s+10)^2\,(s+20)\,s}$$

$$(s+10)^2(s+20)s + 2000 \times 1.4\,(s+5) = 0$$

$$\Rightarrow \quad s^4 + 40s^3 + 500s^2 + 4800s + 14000 = 0$$

$$s_1 \approx -27.5 \qquad s_2 \approx -4.2, \qquad s_{3,4} \approx -4.2 \pm j10.2$$

$$\xi \approx 0.38 \quad \Rightarrow \quad \%OS \approx 27.5 \qquad \omega_n \approx 11.0 \qquad T_s \approx \frac{4}{\xi\omega_n} \approx 0.95$$

%OS a little too large, need to lower k.

$$k = 1.3 \qquad (s+10)^2(s+20)s + 2000 \times 1.3\,(s+5) = 0$$

$$\Rightarrow \quad s^4 + 40s^3 + 500s^2 + 4600s + 13000 = 0$$

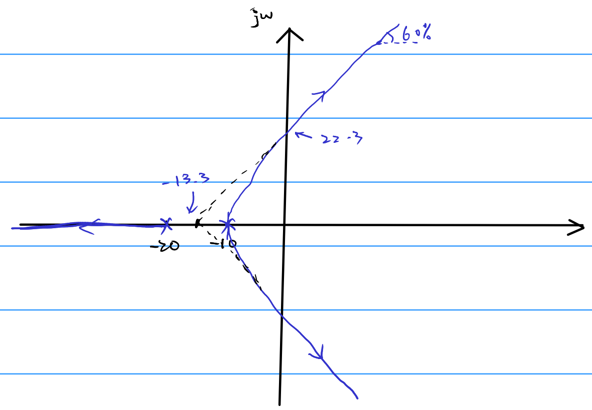
$$s_1 \approx -27.2 \qquad s_2 \approx -4.1, \qquad s_{3,4} \approx -4.34 \pm j9.84$$

$$\xi \approx 0.40 \quad \Rightarrow \quad \%OS \approx 25 \qquad \omega_n \approx 10.75 \qquad T_s \approx \frac{4}{\xi\omega_n} \approx 0.93$$

We can lower k further, based on second-order approximation, but step reponce from MATLAB already shows %OS < 20%

This is because there are two real axis poles, one of them close enough to 0 to lower %OS.

c) Original :



Intersect of asymptotes

$$\frac{-10-10-20}{3} \approx -13.3$$

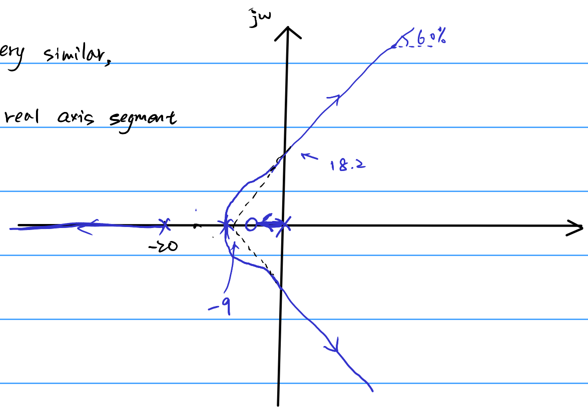
jw crossings

$$\approx \pm j22.3$$

with PI controller :

By design, it looks very similar,

with a new [-5, 0] real axis segment



Differences :

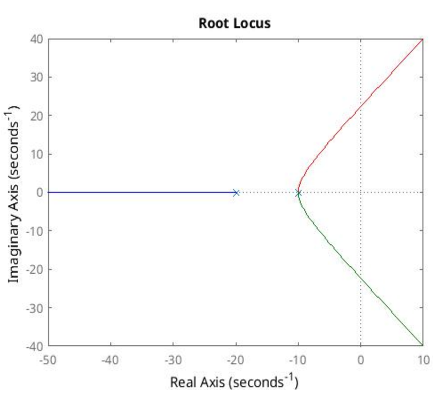
Intersect of asymptotes

$$\frac{-10-10-20-5-0}{5} = -9$$

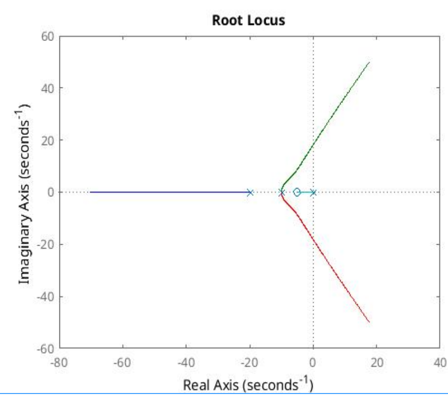
jw crossings $\approx \pm 18.2$

Verify with MATLAB

Original



with PI controller

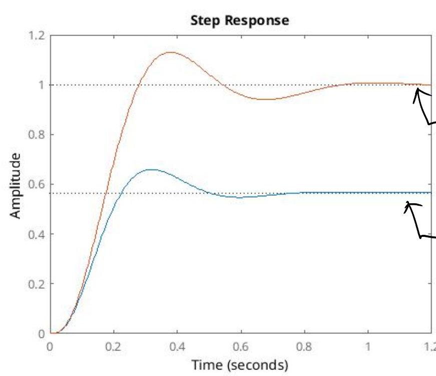


d)

Use MATLAB to compare step responses

With PI control, system SS error is gone.

With the same k, Ts is longer.



PI controller with k=1.3

original with k=1.3

e) For the original system

CL transfer function $T(s) = \frac{2000k}{(s+10)^2(s+20)+2000k} \approx$

$$\lim_{s\to 0} s\cdot\frac{1}{s}\cdot T(s) = \frac{2000k}{2000+2000k} = \frac{k}{1+k}$$

When $k=1.3$ SS error $= 1-\frac{k}{1+k} \approx 0.435$

With PI control. no SS error.

$$\lim_{s\to 0} s\cdot\frac{1}{s}\,\frac{2000k(s+5)}{s(s+10)^2(s+20)+2000k(s+5)} = 1$$

4. (27 pts) PD compensation (Nise 9.3)
Consider open loop plant

$$G(s) = \frac{144K}{s(s+4)(s+12)^2}$$

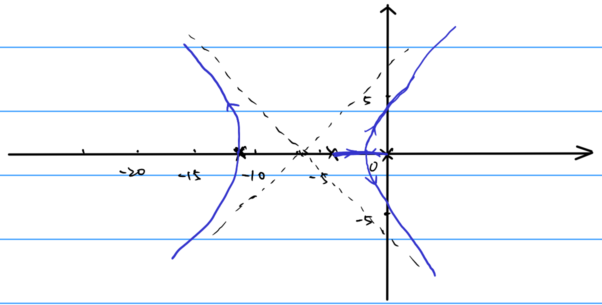
and unity feedback.
[5 pts] a) find K such that overshoot is 20%.
[12 pts] b) Design a PD controller (i.e. find zero location) such that $T_p \approx 0.8$ sec, with the same 20% overshoot.
[6 pts] c) Hand sketch the root locus for the original system and the system with a **PD** compensator, and verify with Matlab.
[2 pts] d) Use Matlab to compare the step response for the closed-loop compensated and uncompensated systems, transient and steady state response.
[2 pts] e) Find the steady state error for a step for both systems.

Solution:

a) First, sketch the RLs 4 poles. 0, −4, −12, −12. Infinite zeros

Asymptotes 45°, 135°, −135°, −45°, intersect at $\frac{0-4-12-12}{4} = -7$

Real segment $[-4, 0]$



Break-away point

$$\frac{1}{\sigma} + \frac{1}{\sigma+4} + \frac{2}{\sigma+12} = 0$$

$$4\sigma^2 + 36\sigma + 48 = 0$$

$\sigma_1 = -7.37$ $\quad$ $\underline{\sigma_2 = -1.63}$

Break-away point for $K>0$

$j\omega$ axis crossings : $\pm j\,4.55$

From Q3, 20% OS corresponds to $\xi \approx 0.456$, approximately 63° from real axis.

Similar to 4. find the dominant poles $(-r, \pm j2r)$ on RLs

$-1.16 \pm j2.32$ $\quad$ $K \approx 8.1 \approx 8$

b) PD controller $G_c(s) = K(s+z)$

Original $T_p = \frac{\pi}{\omega_d} \approx \frac{\pi}{2.32} \approx 1.35\,s$

If we want $T_p' = \frac{\pi}{\omega_d'} = 0.8\,s \Rightarrow \omega_d' \approx 3.93 \approx 4$ with the same %OS

New dominant poles at $-2 \pm j4$

$$\sum_i^{m+n} \theta_i = (2k+1)\pi$$

$$-2\tan^{-1}\left(\frac{4}{12-2}\right) - \tan^{-1}\left(\frac{4}{-2}\right) - \tan^{-1}\left(\frac{4}{4-2}\right) + \tan^{-1}\left(\frac{4}{z-2}\right) = (2k+1)\pi$$

$$\Rightarrow z \approx 6.2$$

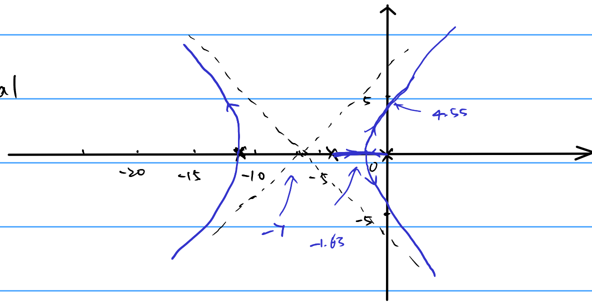
$$|G(p)\cdot G_c(p)\, H(p)| = \frac{1}{K}$$

$$\frac{144^2\,(4.2^2+4^2)}{(2^2+4^2)(2^2+4^2)(10^2+4^2)^2} = \frac{1}{K^2} \Rightarrow K \approx 2.8$$

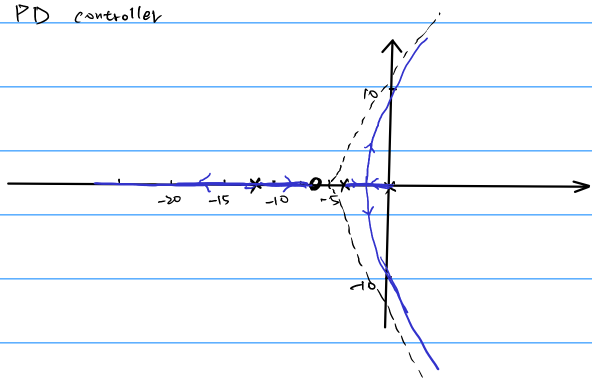
PD controller $2.8(s+6.2)$ (Note. $3(s+6)$ can also work, so do many other approximations)

c)

Original



With PD controller



RLs start at $0, -4, -12, -12$

end at -6, infinity at $\pm 60°$

Asymptotes intersect at $\frac{-4-12-12+6}{3} \approx -7.3$

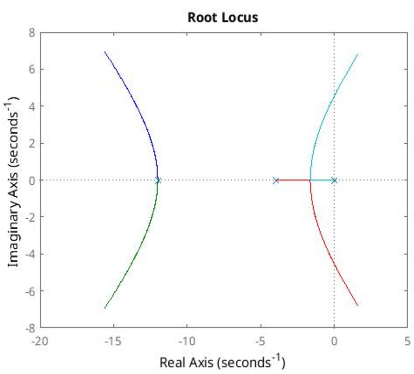
$j\omega$ crossings $\pm j10.2$

Break-away point ≈ -2.2

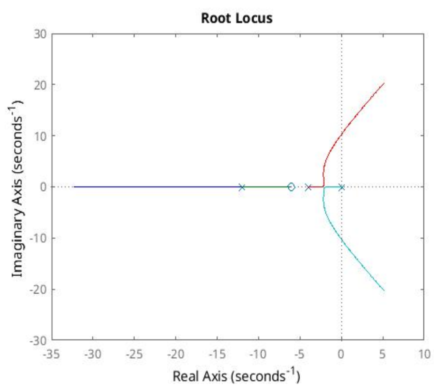
Real segments $[-4, 0]$, $[-12, -6]$ $[-\infty, -12]$

Verify with MATLAB

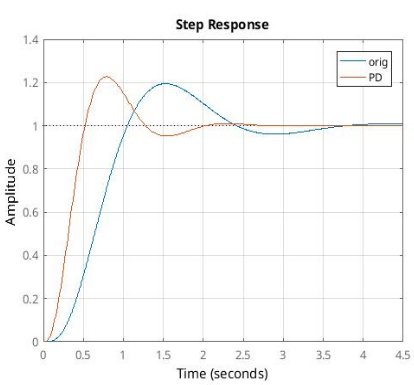
Original



With PD



d)



With PD controller $3(s+6)$

$2.8(s+6.2)$ looks almost identical

Note: %OS of orig and PD are almost the same

But with PD control, settling time is improved,

T_p is reduced.

e) Original

CL transfer function $T(s) = \dfrac{144K}{s(s+4)(s+12)^2 + 144K}$

$\lim_{s \to 0} s \cdot \frac{1}{s} \cdot T(s) = 1$

No SS error.

With PD controller $G(s) \cdot G_c(s) = \dfrac{144k(s+6)}{s(s+4)(s+12)^2}$

CL transfer function $T(s) = \dfrac{144k(s+6)}{s(s+4)(s+12)^2 + 144k(s+6)}$

$\lim_{s \to 0} s \cdot \frac{1}{s} \cdot T(s) = 1$

No SS error.