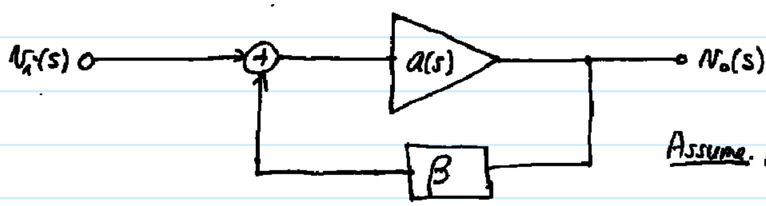
Obtain expressions for overshoot ($V_{overshoot}$) and settling time (T_s) as functions of phase margin, ϕ_m:



Assume β = const. w/ freq.

$$A(s)=\frac{v_o(s)}{v_i(s)}=\frac{a(s)}{1+a(s)\beta}=\frac{a_o\omega_1\omega_2}{s^2+(\omega_1+\omega_2)s+\omega_1\omega_2(1+a_o\beta)}=\underbrace{\frac{A_o\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{A_o\omega_o^2}{s^2+\left(\frac{\omega_o}{Q}\right)s+\omega_o^2}}_{\text{General Lowpass Biquad Transfer Functions}}\qquad[\omega_o=\omega_n]$$

$$\left[a(s)=\frac{a_o}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}=\frac{a_o\omega_1\omega_2}{(s+\omega_1)(s+\omega_2)}\right]$$

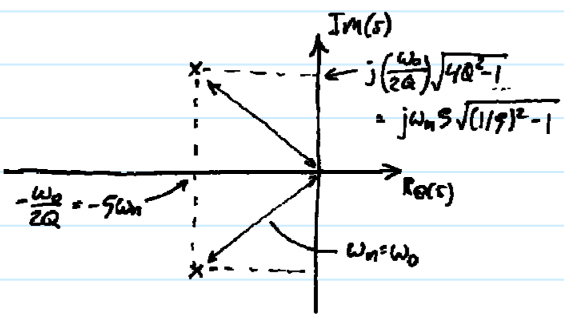
where by direct comparison:

$$\begin{cases}\omega_n=\sqrt{\omega_1\omega_2(1+a_o\beta)}\\ 2\zeta\omega_n=\omega_1+\omega_2\longrightarrow\zeta=\dfrac{\omega_1+\omega_2}{2\omega_n}=\dfrac{1}{2}\dfrac{\omega_1+\omega_2}{\sqrt{\omega_1\omega_2(1+a_o\beta)}}\\ A_o\omega_n^2=a_o\omega_1\omega_2\longrightarrow A_o=\dfrac{a_o}{1+a_o\beta}\end{cases}$$

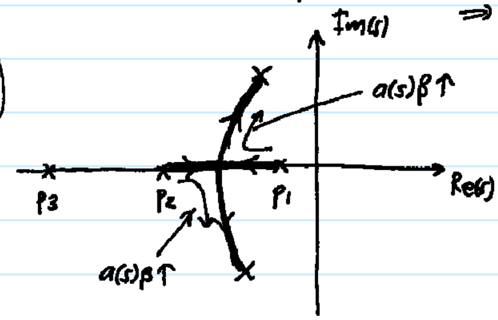
Frequency Response Spectra for various values of ζ on next page

$\boxed{\text{Properties of the General Lowpass Biquad Transfer Function}}$ (a very well-studied function!)

Pole-Zero Diagram:



Root Locus for FB Ckt: poles move as the loop gain increases ⇒ they become complex poles



Time Domain Behavior:

$$v_o(t)=A_oV_i\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\phi\right)\right],\quad\text{where }\phi=\tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

For ζ = small $\rightarrow\phi\simeq\frac{\pi}{2}$

For $\zeta<1$: (for PM < 90°)

$$\%\ \text{Overshoot}=\frac{\text{Peak Value}-\text{Final Value}}{\text{Final Value}}=\exp\left[\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right]\longrightarrow\boxed{V_{overshoot}=V_o\exp\left[\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right]}$$

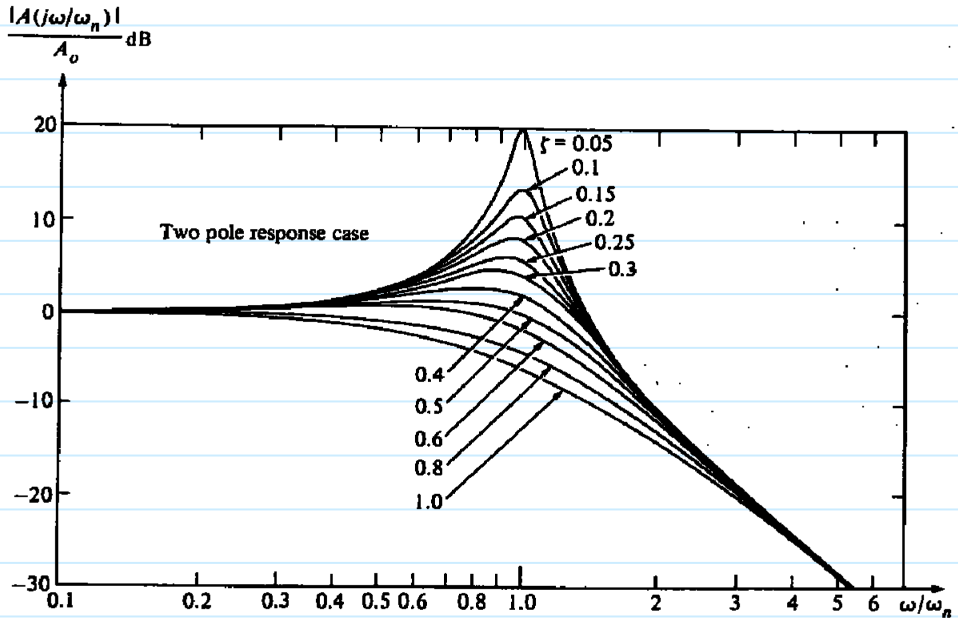
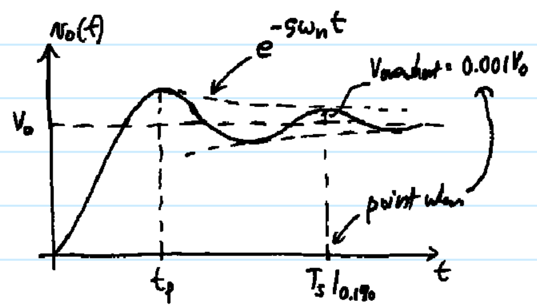


Figure C-2 Gain magnitude response for various values of ζ for a second-order, low-pass system.



Find t_p:

$$\sqrt{1-\zeta^2}\,\omega_n t_p + \phi = \frac{3\pi}{2}$$

$$\left[\zeta = \text{small} \rightarrow \phi \sim \frac{\pi}{2}\right] \Rightarrow \boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

Approximate Determination of 0.1% Settling Time –

$$V_{overshoot}\, e^{-\zeta\omega_n(T_s - t_p)} = 0.001V_o \longrightarrow \boxed{T_s = t_p - \frac{1}{\zeta\omega_n}\ln\left[\frac{0.001V_o}{V_{overshoot}}\right]}$$

$$\longrightarrow \boxed{\zeta = -\frac{1}{\omega_n(T_s - t_p)}\ln\left[\frac{0.001V_o}{V_{overshoot}}\right]}$$

Determine $\phi_m = f(\zeta)$ –

⇒ first get an expression for loop transmission:

$$A(s) = \frac{a(s)}{1 + a(s)\beta} \longrightarrow a(s) = A(s) + a(s)A(s)\beta \longrightarrow a(s) = \frac{A(s)}{1 - \beta A(s)}$$

$$\rightarrow a(s)\beta = \frac{\beta A(s)}{1 - \beta A(s)} = \frac{\beta A_o\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2 - \beta A_o\omega_n^2} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

$\left[\beta \approx \frac{1}{A_o}\right]$

pole @ origin

LHP pole

⇒ get freq. at which $|a(s)\beta| = 1 \rightarrow \omega_u$: $= \omega_{ult}$

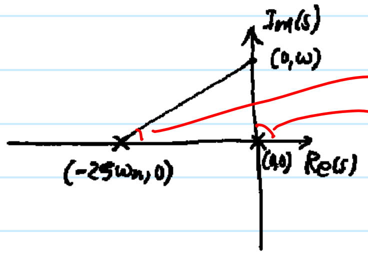
$$a(j\omega)\beta = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega} \Longrightarrow |a(j\omega)\beta| = \frac{\omega_n^2}{\sqrt{\omega^4 + \omega^2 4\zeta^2\omega_n^2}}$$

$$\left[\,|a(j\omega_u)\beta| = 1\,\right] \Longrightarrow \frac{\omega_n^4}{\omega_u^4 + 4\zeta^2\omega_n^2\omega_u^2} = 1 \rightarrow \underline{\omega_u^4 + 4\zeta^2\omega_n^2\omega_u^2 - \omega_n^4 = 0}$$

Solve quadratic: $\boxed{\omega_u = \omega_n\left[\sqrt{4\zeta^2+1} - 2\zeta^2\right]^{1/2}}$

Freq. for phase margin determination.

Find expression for phase:



$$\phi = -\tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{2\zeta\omega_n}\right)$$
$$= -90^\circ - \tan^{-1}\left(\frac{\omega}{2\zeta\omega_n}\right)$$

For Phase Margin: ω_{ult} ω_{ult}

$$\phi_m = 180^\circ + \phi|_{\omega=\omega_u} = 90^\circ - \tan^{-1}\left(\frac{\omega_u}{2\zeta\omega_n}\right)$$

$$\tan(90^\circ - \phi_m) = \frac{\omega_u}{2\zeta\omega_n}$$

$$\left[\tan(A+90^\circ) = -\frac{1}{\tan A}\right] \Longrightarrow -\frac{1}{\tan(-\phi_m)} = \frac{\omega}{2\zeta\omega_n}$$

$$\left[\tan^{-1}(-x) = -\tan^{-1}x\right] \Longrightarrow \boxed{\phi_m = \tan^{-1}\left(\frac{2\zeta\omega_n}{\omega_u}\right) \quad \text{or} \quad \phi_m = \cos^{-1}\left[\sqrt{4\zeta^4+1} - 2\zeta^2\right]}$$

ω_{ult}

Thus:

$$\zeta = \frac{1}{2}\frac{\omega_{ult}}{\omega_n}\tan\phi_m$$

$$\Longrightarrow \boxed{T_s = t_p - \frac{2}{\underset{\omega_{ult}}{\omega_u}\tan\phi_m}\ln\left[\frac{0.001V_o}{V_{overshoot}}\right]}$$

(ignoring effects of slew rate)

⇒ But there's more to it than this. See the following papers:

① B.Y. Kamath, R.G. Meyer, and P.R. Gray, "Relationship between frequency response and settling time of operational amplifiers," IEEE J. of Solid-State Ckts., vol. SC-9, no. 6, pp. 347–352, Dec. 1974.

② R.J. Apfel and P.R. Gray, "A fast-settling monolithic operational amplifier using doublet compression techniques," IEEE J. of Solid-State Ckts, vol. SC-9, no. 6, pp. 332–340, Dec. 1974.