

1. Diagonalization

One of the most powerful ways to think about matrices is to think of them in diagonal form ¹.

- (a) Consider a matrix \mathbf{A} , a matrix \mathbf{V} whose columns are the eigenvectors of \mathbf{A} , and a diagonal matrix $\mathbf{\Lambda}$ with the eigenvalues of \mathbf{A} on the diagonal (in the same order as the eigenvectors (or columns) of \mathbf{V}). From these definitions, show that

$$\mathbf{AV} = \mathbf{V}\mathbf{\Lambda}$$

- (b) We now multiply both sides on the right by \mathbf{V}^{-1} and get $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, the diagonal form of \mathbf{A} . Consider the action of \mathbf{A} on a coordinate vector \vec{x}_u in the standard basis. Interpret each step of the following calculation in terms of coordinate transformations and scaling by eigenvalues.

$$\mathbf{A}\vec{x}_u = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\vec{x}_u$$

2. Matrix Powers

One of the most powerful things about matrix diagonalization is that it gives us some insight into polynomial functions of matrices.

- (a) Write \mathbf{A}^N using the diagonalization of \mathbf{A} and simplify your result as much as possible. What do you get?
- (b) How could you find \mathbf{A} raised to any power while only doing three matrix multiplications?
- (c) Can you suggest an easy way to compute any polynomial function of \mathbf{A} ?

¹Not all matrices can be put in this form but most can. The ones that can't be diagonalized can be put in a similar form called the Jordan form.

3. Fibonacci Sequence

One of the most useful things about diagonalization is that it allows us to easily compute polynomial functions of matrices. This in turn lets us do far more, including solving many linear recurrence relations. This problem shows you how this can be done for the Fibonacci numbers, but you should notice that the same exact technique can be applied far more generally.

- (a) The Fibonacci sequence can be constructed according to the following relation. The N th number in the Fibonacci sequence, F_N , is computed by adding the previous two numbers in the sequence together:

$$F_N = F_{N-1} + F_{N-2}$$

We select the first two numbers in the sequence to be $F_1 = 0$ and $F_2 = 1$ and then we can compute the following numbers as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

Write the operation of computing the next Fibonacci numbers from the previous two using matrix multiplication:

$$\begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} F_{N-1} \\ F_{N-2} \end{bmatrix}$$

- (b) Diagonalize \mathbf{A} to show that

$$F_N = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{N-1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{N-1}$$

is an analytical expression for the N th Fibonacci number.