EECS 16A Fall 2018 — Designing Information Devices and Systems I

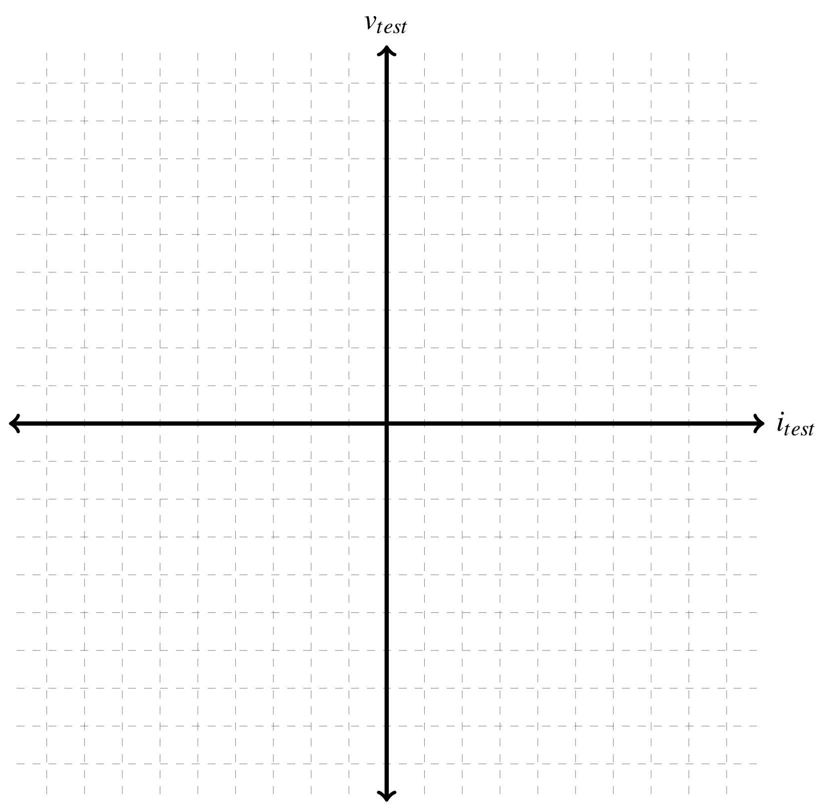
Discussion 13A

1. Ohm's Law With Noise

We are trying to measure the resistance of a black box. We apply various i_{test} currents and measure the ouput voltage v_{test}. Sometimes, we are quite fortunate to get nice numbers. Oftentimes, our measurement tools are a little bit noisy, and the values we get out of them are not accurate. However, if the noise is completely random, then the effect of it can be averaged out over many samples. So we repeat our test many times:

Test	i_{test} (mA)	v_{test} (V)
1	10	21
2	3	7
3	-1	-2
4	5	8
5	-8	-15
6	-5	-11

(a) Plot the measured voltage as a function of the current.



(b) Suppose we stack the currents and voltages to get $\vec{I} = \begin{bmatrix} 10 \\ 3 \\ -1 \\ 5 \\ -8 \\ -5 \end{bmatrix}$ and $\vec{V} = \begin{bmatrix} 21 \\ 7 \\ -2 \\ 8 \\ -15 \\ -11 \end{bmatrix}$. Is there a unique solution for R? What conditions must $\vec{I}$ and $\vec{V}$ satisfy in order for us to solve for R uniquely?

(c) Ideally, we would like to find R such that $\vec{V} = \vec{I}R$. If we cannot do this, we'd like to find a value of R that is the *best* solution possible, in the sense that $\vec{I}R$ is as "close" to $\vec{V}$ as possible. We are defining the sum of squared errors as a **cost function**. In this case the cost function for any value of R quantifies the difference between each component of $\vec{V}$ (i.e. v_j) and each component of $\vec{I}R$ (i.e. i_jR) and sum up the squares of these "differences" as follows:

$$\text{cost}(R) = \sum_{j=1}^{6} (v_j - i_jR)^2$$

Do you think this is a good cost function? Why or why not?

(d) Show that you can also express the above cost function in vector form, that is,

$$\text{cost}(R) = \left\langle (\vec{V} - \vec{I}R), (\vec{V} - \vec{I}R) \right\rangle$$

Hint: $\langle \vec{a}, \vec{b} \rangle = \vec{a}^T \vec{b} = \sum_i a_i b_i$

(e) Find $\hat{R}$, which is defined as the optimal value of R that minimizes cost(R).

Hint: Use calculus. The optimal $\hat{R}$ makes $\frac{d\text{cost}(\hat{R})}{dR} = 0$

(f) On your original *IV* plot, also plot the line $v_{test} = \hat{R}i_{test}$. Can you visually see why this line "fits" the data well? How well would we have done if we had guessed $R = 3\,\text{k}\Omega$? What about $R = 1\,\text{k}\Omega$? Calculate the cost functions for each of these choices of R to validate your answer.

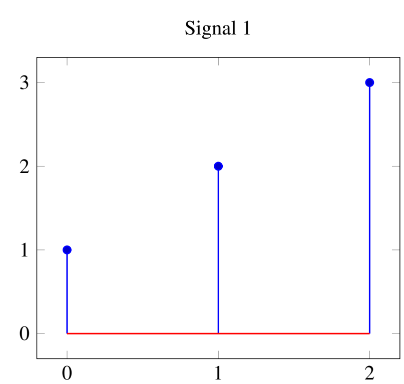
(g) Now, suppose that we add a new data point: $i_7 = 2\,\text{mA}$, $v_7 = 4\,\text{V}$. Will $\hat{R}$ increase, decrease, or remain the same? Why? What does that say about the line $v_{test} = \hat{R}i_{test}$?

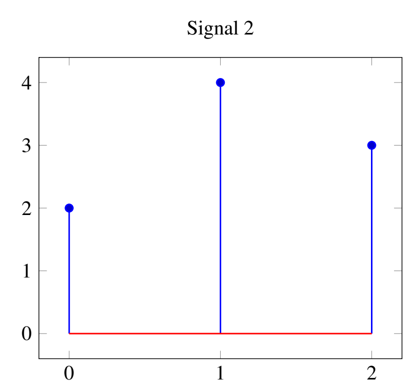
(h) Let's add another data point: $i_8 = 4\,\text{mA}, v_8 = 11\,\text{V}$. Will $\hat{R}$ increase, decrease, or remain the same? Why? What does that say about the line $v_{test} = \hat{R}i_{test}$?

(i) Now your mischievous friend has hidden the black box. You want to predict what output voltage across the terminals if you applied $5.5\,\text{mA}$ through the black box. What would your best guess be?

2. Correlation Revisited

We are given the following two signals, $\vec{s}_1$ and $\vec{s}_2$ respectively:





We have find their cross-correlation for three different scenarios:

- Periodic linear cross-correlation
- Linear cross-correlation when the signals are not periodic
- Circular cross-correlation

Note that periodic linear cross correlation and circular cross correlation are essentially the same — while periodic linear cross correlation looks at linear shifts of an infinite periodic signal, circular correlation looks at circular shifts of one period of the signal.

(a) Find the periodic linear cross correlations, $\text{corr}_{N=3}(\vec{s}_1,\vec{s}_2)$ and $\text{corr}_{N=3}(\vec{s}_2,\vec{s}_1)$ for $\vec{s}_1$ and $\vec{s}_2$, assuming they are periodic with period $N=3$, where periodic linear cross correlation is defined as:

$$\text{corr}_N(\vec{x},\vec{y})[k] = \sum_{i=0}^{N-1} x[i]y[i-k]$$

The signals continue to $+\infty$ and $-\infty$, however, since they are periodic we can focus on just one period of the signal. Calculate the periodic linear cross-correlation assuming the signals are periodic as below (The first one is already done for you!):

$\text{corr}_{N=3}(\vec{s}_1,\vec{s}_2)$

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n]$	2		4		3	
$\langle\vec{s}_1,\vec{s}_2[n]\rangle$	2	+	8	+	9	$=19$

$\text{corr}_{N=3}(\vec{s}_2,\vec{s}_1)$

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n]$						
$\langle\vec{s}_2,\vec{s}_1[n]\rangle$		+		+		=

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n-1]$						
$\langle\vec{s}_1,\vec{s}_2[n-1]\rangle$		+		+		=

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n-1]$						
$\langle\vec{s}_2,\vec{s}_1[n-1]\rangle$		+		+		=

$\vec{s}_1$	1		2		3	
$\vec{s}_2[n-2]$						
$\langle\vec{s}_1,\vec{s}_2[n-2]\rangle$		+		+		=

$\vec{s}_2$	2		4		3	
$\vec{s}_1[n-2]$						
$\langle\vec{s}_2,\vec{s}_1[n-2]\rangle$		+		+		=

(b) Find the two linear cross correlations, $\text{corr}(\vec{s}_1,\vec{s}_2)$ and $\text{corr}(\vec{s}_2,\vec{s}_1)$ for $\vec{s}_1$ and $\vec{s}_2$, assuming they are not periodic, where

$$\text{corr}(\vec{x},\vec{y})[k] = \sum_{i=-\infty}^{\infty} x[i]y[i-k]$$

$\text{corr}(\vec{s}_1,\vec{s}_2)$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n+2]$														
$\langle\vec{s}_1,\vec{s}_2[n+2]\rangle$		+		+		+		+		+		+		=

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n+1]$														
$\langle\vec{s}_1,\vec{s}_2[n+1]\rangle$		+		+		+		+		+		+		=

$\vec{s}_1$	0	0	1	2	3	0	0
$\vec{s}_2[n]$							
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	$+$	$+$	$+$	$+$	$+$	$+$	$=$

$\vec{s}_1$	0	0	1	2	3	0	0
$\vec{s}_2[n-1]$							
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	$+$	$+$	$+$	$+$	$+$	$+$	$=$

$\vec{s}_1$	0	0	1	2	3	0	0
$\vec{s}_2[n-2]$							
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	$+$	$+$	$+$	$+$	$+$	$+$	$=$

$\text{corr}(\vec{s}_2, \vec{s}_1)$

$\vec{s}_2$	0	0	2	4	3	0	0
$\vec{s}_1[n+2]$							
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$	$+$	$+$	$+$	$+$	$+$	$+$	$=$

$\vec{s}_2$	0	0	2	4	3	0	0
$\vec{s}_1[n+1]$							
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$	$+$	$+$	$+$	$+$	$+$	$+$	$=$

$\vec{s}_2$	0	0	2	4	3	0	0
$\vec{s}_1[n]$							
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	$+$	$+$	$+$	$+$	$+$	$+$	$=$

$\vec{s}_2$	0	0	2	4	3	0	0
$\vec{s}_1[n-1]$							
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	$+$	$+$	$+$	$+$	$+$	$+$	$=$

$\vec{s}_2$	0	0	2	4	3	0	0
$\vec{s}_1[n-2]$							
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	$+$	$+$	$+$	$+$	$+$	$+$	$=$

(c) Find the circular cross correlation, $\text{circcorr}(\vec{s}_1, \vec{s}_2)$ and $\text{circcorr}(\vec{s}_2, \vec{s}_1)$ for $\vec{s}_1$ and $\vec{s}_2$, where

$$\text{circcorr}(\vec{x}, \vec{y})[k] = \sum_{i=0}^{N-1} x[i]y[i-k]_N$$