EECS 16A Designing Information Devices and Systems I
Fall 2018 Discussion 14A

1. Orthonormal Matrices and Projections

An orthonormal matrix, $\mathbf{A}$, is a matrix whose columns, $\vec{a}_i$, are:

- Orthogonal (ie. $\langle \vec{a}_i, \vec{a}_j \rangle = 0$ when $i \neq j$)
- Normalized (ie. vectors with length equal to 1, $\|\vec{a}_i\| = 1$). This implies that $\|\vec{a}_i\|_2 = \langle \vec{a}_i, \vec{a}_i \rangle = 1$.

(a) Suppose that the matrix $\mathbf{A} \in \mathbb{R}^{N \times M}$ has linearly indpendent columns. The vector $\vec{y}$ in $\mathbb{R}^N$ is not in the subspace spanned by the columns of $\mathbf{A}$. What is the projection of $\vec{y}$ onto the subspace spanned by the columns of $\mathbf{A}$?

(b) Show if $\mathbf{A} \in \mathbb{R}^{N \times N}$ is an orthonormal matrix then the columns, $\vec{a}_i$, form a basis for $\mathbb{R}^N$.

(c) When $\mathbf{A} \in \mathbb{R}^{N \times M}$ and $N \geq M$ (i.e. tall matrices), show that if the matrix is orthonormal, then $\mathbf{A}^T\mathbf{A} = \mathbf{I}_{M \times M}$.

(d) Again, suppose $\mathbf{A} \in \mathbb{R}^{N \times M}$ where $N \geq M$ is an orthonormal matrix. Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of $\mathbf{A}$ is now $\mathbf{A}\mathbf{A}^T\vec{y}$.

2. Orthogonal Matching Pursuit

Let's work through an example of the OMP algorithm. Suppose that we have a vector $\vec{x} \in \mathbb{R}^4$ that is sparse and we know that it has only 2 non-zero entries. In particular,

$$\mathbf{M}\vec{x} \approx \vec{y} \tag{1}$$

$$\begin{bmatrix} | & | & | & | \\ \vec{m}_1 & \vec{m}_2 & \vec{m}_3 & \vec{m}_4 \\ | & | & | & | \end{bmatrix} \vec{x} \approx \vec{y} \tag{2}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \tag{3}$$

where exactly 2 of x_1 to x_4 are non-zero. Use Orthogonal Matching Pursuit to estimate x_1 to x_4.

(a) Why can we not solve for $\vec{x}$ directly?

(b) Why can we not apply the least squares process to obtain $\vec{x}$?

(c) Let us start by reviewing the OMP procedure,

Inputs:

- A matrix $\mathbf{M}$, whose columns, $\vec{m}_i$, make up a set of vectors, $\{\vec{m}_i\}$, each of length n
- A vector $\vec{y}$ of length n
- The sparsity level k of the signal

Outputs:

- A vector $\vec{x}$, that contains k non-zero entries.
- A error vector $\vec{e} = \vec{y} - \mathbf{M}\vec{x}$

Procedure:

- Initialize the following values: $\vec{e} = \vec{y}$, $j = 1$, k, $\mathbf{A} = \begin{bmatrix} \end{bmatrix}$
- while ($j \leq k$):
  i. Compute the inner product for each vector in the set, $\vec{m}_i$, with $\vec{e}$: $c_i = \langle \vec{m}_i, \vec{e} \rangle$.
  ii. Column concatenate matrix $\mathbf{A}$ with the column vector that had the maximum inner product value with $\vec{e}$, c_i: $\mathbf{A} = \begin{bmatrix} \mathbf{A} & | & \vec{m}_i \end{bmatrix}$
  iii. Use least squares to compute $\vec{x}$ given the $\mathbf{A}$ for this iteration: $\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$
  iv. Update the error vector: $\vec{e} = \vec{y} - \mathbf{A}\vec{x}$
  v. Update the counter: $j = j + 1$

(d) Compute the inner product of every column with the $\vec{y}$ vector. Which column has the largest inner product? This will be the first column of the matrix $\mathbf{A}$.

(e) Now, find the projection of $\vec{y}$ onto the columns of $\mathbf{A}$ (ie. $\text{proj}_{\text{Col}(\mathbf{A})}\vec{y} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$). Use this to update the error vector.

(f) Now compute the inner product of every column with the new error vector. Which column has the largest inner product? This will be the second column of $\mathbf{A}$.

(g) We now have two non-zero entries for our vector, $\vec{x}$. Find the values of those two entries.

(Reminder: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)