EECS 16A Designing Information Devices and Systems I
Fall 2018 Discussion 2B

1. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices!

Part 1: Rotation Matrices as Rotations

(a) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by 15° and 30°, respectively. Design a procedure to rotate the unit square by 45° using only $\mathbf{T}_1$ and $\mathbf{T}_2$, and plot the result in the IPython notebook. How would you rotate the square by 60°?

(b) Try to rotate the unit square by 60° using only one matrix. What does this matrix look like?

(c) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. (*Hint: Use your trigonometric identities!*)

(d) Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? *Don't use inverses!* (**Note:** We do not expect you to know inverses at this point; we will cover them soon.)

(e) Use part (d) to obtain the "inverse" rotation matrix for a matrix that rotates a vector by θ. Multiply the inverse rotation matrix with the rotation matrix and vice-versa. What do you get?

[Practice Problem] Part 2: Commutativity of Operations

A natural question to ask is the following: Does the *order* in which you apply these operations matter? Follow your TA to obtain the answers to the following questions!

(a) Let's see what happens to the unit square when we rotate the square by 60° and then reflect it along the y-axis.

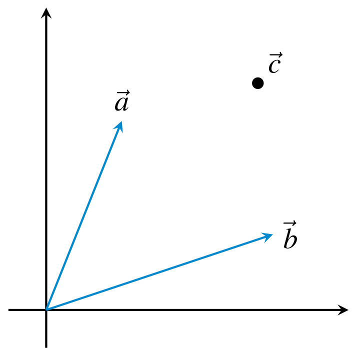
(b) Now, let's see what happens to the unit square when we first reflect the square along the y-axis and then rotate it by 60°.

(c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

2. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



(a) First, consider the case where $\vec{a}=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{b}=\begin{bmatrix}1\\1\end{bmatrix}$, and $\vec{c}=\begin{bmatrix}-2\\2\end{bmatrix}$. Draw these vectors on a sheet of paper. Now find the two scalars α and β, such that we reach point $\vec{c}$. What are these scalars if we use $\vec{a}=\begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}2\\1\end{bmatrix}$ instead?

(b) Now formulate the general problem as a system of linear equations and write it in matrix form.

3. Proofs

(a) **[Practice Problem]:** Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x}=\vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(b) **[Practice Problem]:** Suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x}=\vec{b}$, that is, $\mathbf{A}\vec{x}_1=\vec{b}$ and $\mathbf{A}\vec{x}_2=\vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(c) Suppose there exists a matrix $\mathbf{A}$ whose columns are linearly dependent. Prove that if there exists a solution to $\mathbf{A}\vec{x}=\vec{b}$, then there are infinitely many solutions.

4. Matrix Multiplication

Consider the following matrices:

$$\mathbf{A}=\begin{bmatrix}1&4\\2&3\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}3&2\\2&1\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}1&9&5&7\\4&3&2&2\end{bmatrix}\quad \mathbf{D}=\begin{bmatrix}5&5&8\\6&1&2\\4&1&7\\3&2&2\end{bmatrix}\quad \mathbf{E}=\begin{bmatrix}8&1&6\\3&5&7\\4&9&2\end{bmatrix}\quad \mathbf{F}=\begin{bmatrix}5&3&4\\1&8&2\\2&3&5\end{bmatrix}$$

For each matrix multiplication problem, if the product exists, find the product by hand. Otherwise, explain why the product does not exist.

(a) **AB**
(b) **BA**
(c) **CD**
(d) **DC**
(e) **EF**
(f) **FE**