EECS 16A Designing Information Devices and Systems I
Fall 2018 Discussion 3B

1. Commutativity of Operations

You've learned about matrices as transformations, and so a question that we might have is: Does the *order* in which you apply operations matter? We'll be working with a unit square as an object we're going to transform. Follow your TA to obtain the answers to the following questions!

(a) Let's see what happens to the unit square when we rotate the square by 60° and then reflect it along the y-axis.

(b) Now, let's see what happens to the unit square when we first reflect the square along the y-axis and then rotate it by 60°.

(c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

2. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

(a)

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

(b)

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$$

In other words, we can swap the order of our spanning vectors and not change their span.

(c) **[Practice Problem]:**

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

3. Proofs

(a) **[Practice Problem]:** Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(b) **[Practice Problem]:** Suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(c) Suppose there exists a matrix $\mathbf{A}$ whose columns are linearly dependent. Prove that if there exists a solution to $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

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Reference Definitions

Vector spaces: A *vector space* V is a set of elements that is 'closed' under vector addition and scalar multiplication and contains a zero vector. What does closed mean?

That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

More formally, a *vector space* (V, F) is a set of vectors V, a set of scalars F, and two operators that satisfy the following properties:

- Vector Addition
  - Associative: $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ for any $\vec{v},\vec{u},\vec{w} \in V$.
  - Commutative: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ for any $\vec{v},\vec{u} \in V$.
  - Additive Identity: There exists an additive identity $\vec{0} \in V$ such that $\vec{v}+\vec{0} = \vec{v}$ for any $\vec{v} \in V$.
  - Additive Inverse: For any $\vec{v} \in V$, there exists $-\vec{v} \in V$ such that $\vec{v}+(-\vec{v}) = \vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for any $\vec{v} \in V$, $\alpha,\beta \in F$.
  - Multiplicative Identity: There exists $1 \in F$ where $1 \cdot \vec{v} = \vec{v}$ for any $\vec{v} \in F$. We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$ for any $\alpha \in F$ and $\vec{u},\vec{v} \in V$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v}$ for any $\alpha,\beta \in F$ and $\vec{v} \in V$.

Subspaces: A subset W of a *vector space* V is a *subspace* of V if the above conditions (closure under vector addition and scalar multiplication and existence of a zero vector) hold for the elements in the subspace W.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as their subspaces.

4. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \,\middle|\, \vec{v} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix}, \text{ where } c,d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

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