1. Identifying a Basis

Does each of these sets of vectors describe a basis for $\mathbb{R}^3$? What about for some subspace of $\mathbb{R}^3$?

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

2. Constructing a Basis

Let’s consider a subspace of $\mathbb{R}^3$ called $V$ which has the following property: for every vector in $V$, the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$, then $a_1 = 2(a_2 + a_3)$.

Find a basis for $V$. What is the dimension of $V$?

3. Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let’s consider the vector space $\mathbb{R}^m$ and a set of $n$ vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$.

(a) For the first part of the problem, let $m > n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

(b) Let $m = n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

(c) Now, let $m < n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? What vector space could they form a basis for?

Hint: Think about whether the vectors can be linearly independent.

4. Exploring Column Spaces and Null Spaces

- The column space is the possible outputs of a transformation/function/linear operation. It is also the span of the column vectors of the matrix.
- The null space is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $A$? What is its dimension?

ii. What is the null space of $A$? What is its dimension?

iii. Are the column spaces of the row reduced matrix $A$ and the original matrix $A$ the same?

iv. Do the columns of $A$ form a basis for $\mathbb{R}^2$ (or $\mathbb{R}^3$ for part (b))? Why or why not?
5. Inverse Proof

Prove that a matrix $A$ is invertible if and only if its columns are linearly independent.