EECS 16A Designing Information Devices and Systems I
Fall 2018 Discussion 4B

Reference Definitions: Orthogonality We've seen that the following statements are equivalent for an $n \times n$ matrix $\mathbf{A}$:

- $\mathbf{A}$ is invertible
- The equation $\mathbf{A}\vec{x} = \vec{0}$, has a unique solution, which is $\vec{x} = \vec{0}$
- The columns of $\mathbf{A}$ are linearly independent
- For each column vector $\vec{b} \in \mathbb{R}^n$, $\mathbf{A}\vec{x} = \vec{b}$ has a unique solution $\vec{x}$
- Null($\mathbf{A}$) $= \vec{0}$

Conversely, the opposites are equivalent statements:

- $\mathbf{A}$ is not invertible
- $\mathbf{A}\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$
- The columns of $\mathbf{A}$ are linearly dependent
- There is not be a unique $\vec{x}$ for every $\vec{b}$ where $\mathbf{A}\vec{x} = \vec{b}$
- Null($\mathbf{A}$) contains more than just the zero vector $\vec{0}$

These are part of what is known as the Invertible Matrix Theorem.

1. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

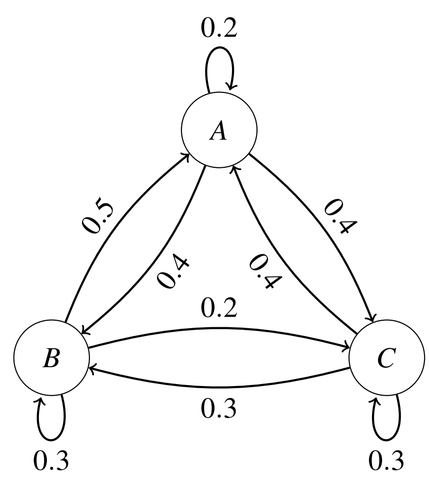

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix representing the pumps system.

(b) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129, B_0 = 109, C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

Hint: If $\vec{x}_{ss} = \begin{bmatrix} A_{ss} \\ B_{ss} \\ C_{ss} \end{bmatrix}$ is a vector describing the steady state levels of water in the reservoirs (in kiloliters), what happens if you fill the reservoirs A, B and C with A_{ss}, B_{ss} and C_{ss} kiloliters of water, respectively, and apply the pumps once?

Hint II: Note that the pumps system preserves the total amount of water in the reservoirs. That is, no water is lost or gained by applying the pumps.

2. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and the associated eigenvectors.

(a) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

(b) $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

(c) **(PRACTICE) M =** $\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$

(d) **(PRACTICE) M =** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

3. Mechanical Determinants

(a) Compute the determinant of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(b) Compute the determinant of $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

(c) Compute the determinant of $\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

4. Row Operations and Determinants

In this question we explore the effect of row operations on the determinant of a matrix. Note that scaling a row by a will increase the determinant by a factor of a, and adding a multiple of one row to another will not change the determinant. Swapping two rows of a matrix and computing the determinant is equivalent to multiplying the determinant of the original matrix by -1. The determinant of an identity matrix is 1. Feel free to prove these properties to convince yourself that they hold for general square matrices.

(a) An upper triangular matrix is a matrix with zeros below its diagonal. For example a 3×3 upper triangular matrix is:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{bmatrix}$$

By considering row operations and what they do to the determinant, argue that the determinant of a general $n \times n$ upper triangular matrix is the product of its diagonal entries if they are non-zero. For example, the determinant of the 3×3 matrix above is $a_1 \cdot b_2 \cdot c_3$ if $a_1, b_2, c_3 \neq 0$.

(b) If the diagonal of an upper-triangular matrix has a zero entry, argue that its determinant is still the product of its diagonal entries.

5. (PRACTICE) StateRank Car Rentals

You are an analyst at StateRank Car Rentals, which operates in California, Oregon, and Nevada. You are hired to analyze the number of rental cars going into and out of each of the three states (CA, OR, and NV). The number of cars in each state on day $n \in \{0, 1, \ldots\}$ can be represented by the state vector $\vec{s}[n] = \begin{bmatrix} s_{\text{CA}}[n] \\ s_{\text{OR}}[n] \\ s_{\text{NV}}[n] \end{bmatrix}$.

The state vector follows the state evolution equation $\vec{s}[n+1] = \mathbf{A}\vec{s}[n], \forall n \in \{0, 1, \ldots\}$, where the transition matrix, $\mathbf{A}$, of this linear dynamic system is

$$\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$$

(a) Use the designated boxes in Figure 2 to **fill in the weights for the daily travel dynamics of rental cars between the three states**, as described by the state transition matrix $\mathbf{A}$.

Note the order of the elements in the state vector $\vec{s}[n]$.

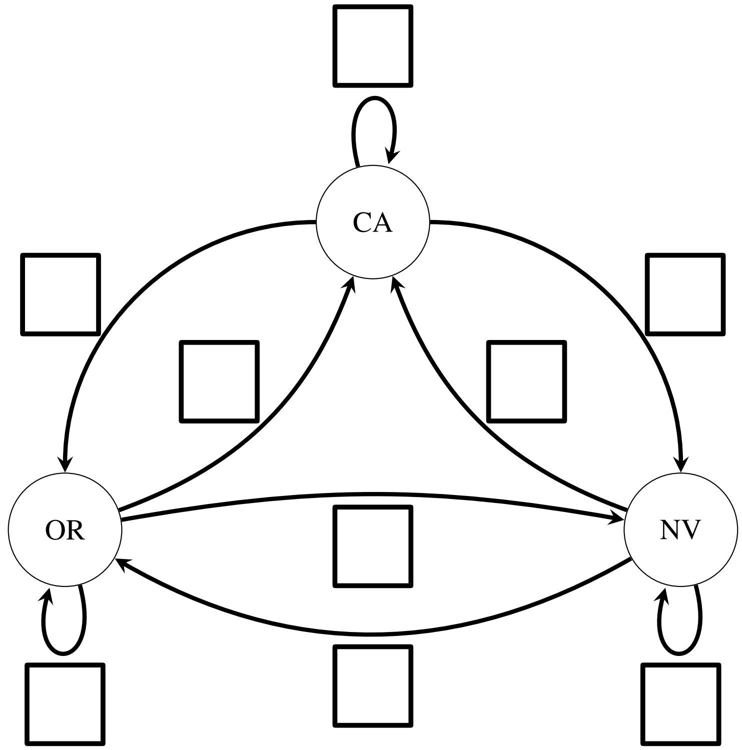

Figure 2: StateRank Rental Cars Daily Travel Dynamics.

A copy of the state transition matrix for reference: $\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$

(b) Suppose the state vector on day $n = 4$ is $\vec{s}[4] = \begin{bmatrix} 100 \\ 200 \\ 100 \end{bmatrix}$. **Calculate the state vector on day 5, $\vec{s}[5]$.**

(c) We want to express the number of cars in each state on day n as a function of the initial number of cars in each state on day 0. That is, we write $\vec{s}[n]$ in terms of $\vec{s}[0]$ as follows:

$$\vec{s}[n] = \mathbf{B}\vec{s}[0]$$

Express the matrix B in terms of A and n.

A copy of the state transition matrix for reference: $\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$

(d) We denote the eigenvalue/eigenvector pairs of the matrix $\mathbf{A}$ by

$$\left(\lambda_1 = 1, \vec{u}_1 = \begin{bmatrix} 50 \\ 40 \\ 110 \end{bmatrix}\right), \left(\lambda_2, \vec{u}_2 = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}\right), \text{ and } \left(\lambda_3, \vec{u}_3 = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}\right).$$

Find the eigenvalues λ_2 and λ_3 corresponding to the eigenvectors $\vec{u}_2$ and $\vec{u}_3$, respectively. Note that since $\lambda_1 = 1$ is given, you don't have to calculate it.

(e) For the given dynamics in this problem, **does a matrix C exists such that $\vec{s}[n-1] = \mathbf{C}\vec{s}[n]$, for $n \in \{1, 2, \ldots\}$? Justify your answer.**

A copy of the state transition matrix for reference: $\mathbf{A} = \begin{bmatrix} 7/10 & 1/10 & 1/10 \\ 1/10 & 6/10 & 1/10 \\ 2/10 & 3/10 & 8/10 \end{bmatrix}$

(f) Suppose that the initial number of rental cars in each state on day 0 is

$$\vec{s}[0] = \begin{bmatrix} 7000 \\ 5000 \\ 8000 \end{bmatrix} = 100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3,$$

where $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ are the eigenvectors from part (d).
After a very large number of days n, how many rental cars will there be in each state?
That is, i) calculate

$$\vec{s}^* = \lim_{n\to\infty} \vec{s}[n]$$

<u>and</u> ii) show that the system will indeed converge to $\vec{s}^*$ as $n \to \infty$ if it starts from $\vec{s}[0]$.
Hint: If you didn't solve part (d), the eigenvalues satisfy $\lambda_1 = 1, |\lambda_2| < 1$ and $|\lambda_3| < 1$.