Consider a vector in the standard basis, 
\[
\vec{x} = a\vec{e}_1 + b\vec{e}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = I\vec{x}
\] (1)
where, \(a, b\) are \(\vec{x}\)’s coordinates in the standard basis.

Given a new set of basis vectors, \(\mathcal{V} = \{\vec{v}_1, \vec{v}_2\}\), if \(\vec{x} \in \text{span}\{\mathcal{V}\}\), then we can find new coordinates in terms of this new basis. The new coordinates are called \(a_v, b_v\) and are described,
\[
\vec{x} = a_v\vec{v}_1 + b_v\vec{v}_2 = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix} = \mathcal{V}\vec{x}_v
\] (2)

Now consider another set of basis vectors, \(\mathcal{U} = \{\vec{u}_1, \vec{u}_2\}\), if \(\vec{x} \in \text{span}\{\mathcal{U}\}\), then we can find the coordinates in terms of this basis. These coordinates are called \(a_u, b_u\) and are described,
\[
\vec{x} = a_u\vec{u}_1 + b_u\vec{u}_2 = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = \mathcal{U}\vec{x}_u
\] (3)

All of these bases are equivalent representations of any vector \(\vec{x} \in \mathbb{R}^2\); each with their own set of coordinates.
\[
\vec{x} = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix}
\] (4)

\[
\vec{x} = I\vec{x} = \mathcal{V}\vec{x}_v = \mathcal{U}\vec{x}_u
\] (5)

1. Coordinate Change Examples

(a) **Transformation From Standard Basis To Another Basis in \(\mathbb{R}^3\)**

Calculate the coordinate transformation between the following bases
\[
\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},
\]
i.e. find a matrix \(\mathbf{T}\), such that \(\vec{x}_v = \mathbf{T}\vec{x}_u\) where \(\vec{x}_u\) contains the coordinates of a vector in a basis of the columns of \(\mathbf{U}\) and \(\vec{x}_v\) is the coordinates of the same vector in the basis of the columns of \(\mathbf{V}\).

Let \(\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\) and compute \(\vec{x}_v\). Repeat this for \(\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\). Now let \(\vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\). What is \(\vec{x}_v\)?
(b) Transformation Between Two Bases in $\mathbb{R}^3$

Calculate the coordinate transformation between the following bases

\[
U = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},
\]

i.e. find a matrix $T$, such that $\vec{x}_v = T\vec{x}_u$. Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_v$. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Now let $\vec{x}_u = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{x}_v$?

(c) What is the coordinate transformation from $\vec{x}_v$ to $\vec{x}_u$, i.e. find $W$ such $\vec{x}_u = W\vec{x}_v$?

(d) Transformation Between General Bases in $\mathbb{R}^2$

Calculate the coordinate transformation between the following bases

\[
U = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},
\]

i.e. find a matrix $T$, such that $\vec{x}_v = T\vec{x}_u$. Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and compute $\vec{x}_v$. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now let $\vec{x}_u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. What is $\vec{x}_v$?

2. Proofs

(a) Let $A$ be an invertible matrix. Show that if $\lambda$ is an eigenvalue of $A$, then $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$.

3. Steady and Unsteady States

(a) You’re given the matrix $M$ (below) which describes some physical system (could describe either people or water):

\[
M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}
\]

Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$

ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$

iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$. The values $\alpha, \beta, \gamma$ are the coordinates for the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. For each of the cases in the table, determine if

$$\lim_{n \to \infty} M^n \vec{x}$$

converges. If it does, what does it converge to?
$\begin{array}{ccc|c} \alpha & \beta & \gamma & \text{Converges?} \\ \hline 0 & 0 & \neq 0 & \hline 0 & \neq 0 & 0 & \hline 0 & \neq 0 & \neq 0 & \hline \neq 0 & 0 & 0 & \hline \neq 0 & 0 & \neq 0 & \hline \neq 0 & \neq 0 & 0 & \hline \neq 0 & \neq 0 & \neq 0 & \end{array}$

### 4. More Practice with Column Spaces and Null Spaces

- The **column space** is the possible outputs of a transformation/function/linear operation. It is also the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $A$? What is its dimension?

ii. What is the null space of $A$? What is its dimension?

iii. Are the column spaces of the row reduced matrix $A$ and the original matrix $A$ the same?

iv. Do the columns of $A$ form a basis for $\mathbb{R}^2$ (or $\mathbb{R}^3$ for part (b))? Why or why not?

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$