EECS 16A Designing Information Devices and Systems I
Fall 2019 Discussion 12A

1. **Reference**

   Inner products:

   The inner product is for two vectors $\vec{x}$ and $\vec{y}$ defined as:

   $$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}$$

   The inner product satisfies the following properties:

   (a) **Symmetry:** $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.

   (b) **Linearity:**

      i. $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$.

      ii. $\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$.

   (c) **Norm:** $\langle \vec{x}, \vec{x} \rangle = \| \vec{x} \|^2$.

   (d) If $\langle \vec{x}, \vec{x} \rangle = 0$, then $\vec{x} = 0$.

   Cauchy-Schwartz inequality:

   Recall that

   $$\langle \vec{v}, \vec{w} \rangle = \| \vec{v} \| \cdot \| \vec{w} \| \cos\theta,$$

   where θ is the angle between the two vectors $\vec{v}$ and $\vec{w}$. This leads to the Cauchy-Schwarz inequality, which states that for two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$:

   $$\langle \vec{v}, \vec{w} \rangle \leq \| \vec{v} \| \cdot \| \vec{w} \| .$$

   Cross-correlation:

   The cross-correlation between two signals $r[n]$ and $s[n]$ is defined as follows:

   $$\text{corr}_r(s)[k] = \sum_{i=-\infty}^{\infty} r[i]s[i-k].$$

2. **A review of Inner Products**

   Find the inner product of the following three pairs of vectors.

   (a)

   $$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   Answer: Recall that the inner product of two vectors $\vec{x}$ and $\vec{y}$ is $\vec{x}^T \vec{y}$, thus:

   $$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 3 = 4$$

(b)

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$$

Answer: When working with real numbers, the inner product is commutative, thus the answer is the same as part a), 4

(c)

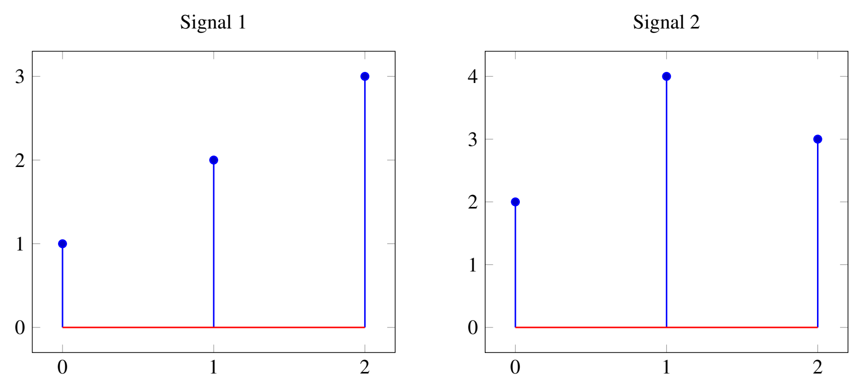
$$\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}-3\\2\\1\end{bmatrix}$$

Answer:

$$\begin{bmatrix}1 & 0 & 3\end{bmatrix}\begin{bmatrix}-3\\2\\1\end{bmatrix} = -3+3 = 0$$

3. Correlation

We are given the following two signals, $s_1[n]$ and $s_2[n]$ respectively.



Find the cross correlations, $\text{corr}_{s_1}(s_2)$ and $\text{corr}_{s_2}(s_1)$ for signals $s_[n]$ and $s_2[n]$. Recall

$$\text{corr}_x(y)[k] = \sum_{i=-\infty}^{\infty} x[i]y[i-k].$$

$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n+2]$														
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$		+		+		+		+		+		+		=

$\vec{s}_1$	0		0		1		2		3		0		0	
$\vec{s}_2[n+1]$														
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$		+		+		+		+		+		+		=

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n]$								
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	+	+	+	+	+	+	=	

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n-1]$								
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	+	+	+	+	+	+	=	

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n-2]$								
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	+	+	+	+	+	+	=	

$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$

$\vec{s}_2$	0	0	2	4	3	0	0	
$\vec{s}_1[n+2]$								
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$	+	+	+	+	+	+	=	

$\vec{s}_2$	0	0	2	4	3	0	0	
$\vec{s}_1[n+1]$								
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$	+	+	+	+	+	+	=	

$\vec{s}_2$	0	0	2	4	3	0	0	
$\vec{s}_1[n]$								
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	+	+	+	+	+	+	=	

$\vec{s}_2$	0	0	2	4	3	0	0	
$\vec{s}_1[n-1]$								
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	+	+	+	+	+	+	=	

$\vec{s}_2$	0	0	2	4	3	0	0	
$\vec{s}_1[n-2]$								
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	+	+	+	+	+	+	=	

Answer: The linear cross-correlation is calculated by shifting the second signal both forward and backward until there is no overlap between the signals. When there is no overlap, the cross-correlation goes to zero. Both of these cross-correlations should have only zeros outside the range: $-2 \leq n \leq 2$.

$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n+2]$	2	4	3	0	0	0	0	
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$	0 +	0 +	3 +	0 +	0 +	0 +	0	$= 3$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n+1]$	0	2	4	3	0	0	0	
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$	0 +	0 +	4 +	6 +	0 +	0 +	0	$= 10$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	0 +	0 +	2 +	8 +	9 +	0 +	0	$= 19$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n-1]$	0	0	0	2	4	3	0	
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	0 +	0 +	0 +	4 +	12 +	0 +	0	$= 16$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n-2]$	0	0	0	0	2	4	3	
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	0 +	0 +	0 +	0 +	6 +	0 +	0	$= 6$

$$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$$

$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\vec{s}_1[n+2]$	1	2	3	0	0	0	0	
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$	0 +	0 +	6 +	0 +	0 +	0 +	0	$= 6$

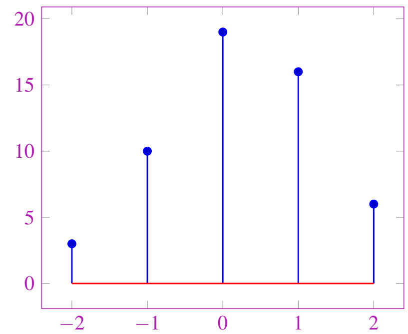
$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\vec{s}_1[n+1]$	0	1	2	3	0	0	0	
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$	0 +	0 +	4 +	12 +	0 +	0 +	0	$= 16$

$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\vec{s}_1[n]$	0	0	1	2	3	0	0	
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	0 +	0 +	2 +	8 +	9 +	0 +	0	$= 19$

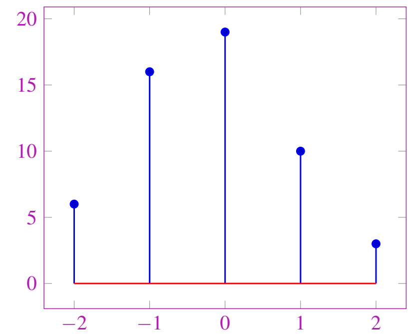
$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\vec{s}_2[n-1]$	0	0	0	1	2	3	0	
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	0 +	0 +	0 +	4 +	6 +	0 +	0	$= 10$

$\vec{s}_2[n]$	0		0		2		4		3		0		0	
$\vec{s}_2[n-2]$	0		0		0		0		1		2		3	
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	0	+	0	+	0	+	0	+	3	+	0	+	0	$=3$

Cross-correlation of $\vec{s}_1$ and $\vec{s}_2$



Cross-correlation of $\vec{s}_2$ and $\vec{s}_1$

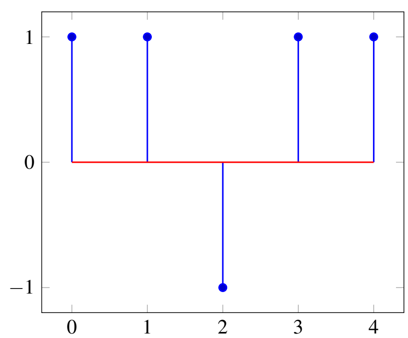


Notice that $\text{corr}_{\vec{s}_1}(\vec{s}_2)[k] = \text{corr}_{\vec{s}_2}(\vec{s}_1)[-k]$, i.e. changing the order of the signals reverses the cross-correlation sequence.

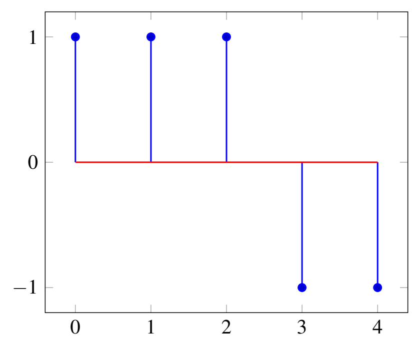
4. Identifying satelites and their delays

We are given the following two signals, $\vec{s}_1$ and $\vec{s}_2$ respectively, that are signatures for two satellites.

Signal 1

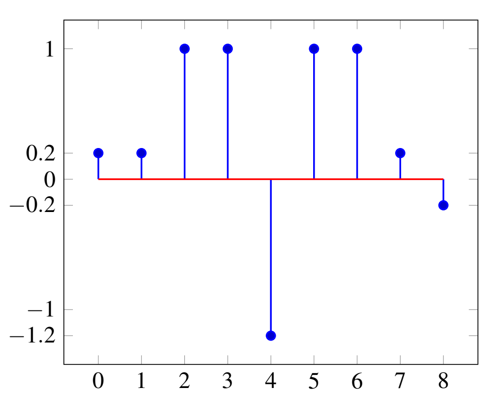


Signal 2



(a) Your cellphone antenna receives the following signal $r[n]$. You know that there may be some noise present in $r[n]$ in addition to the transmission from the satellite.

$\vec{r}$



Which satellites are transmitting? What is the delay between the satellite and your cellphone? Use cross-correlation to justify your answer.

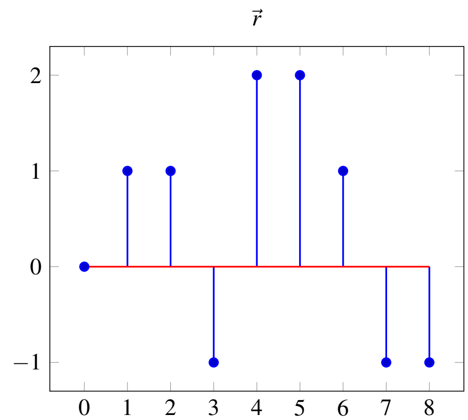
Answer: We calculate both $\text{corr}_r(\vec{s_1})[k]$ and $\text{corr}_r(\vec{s_2})[k]$:

$$
\begin{aligned}
\text{corr}_{\vec{r}}(\vec{s}_1)[-4] &= (0.2)(1) = 0.2\\
\text{corr}_{\vec{r}}(\vec{s}_1)[-3] &= (0.2)(1)+(0.2)(1) = 0.4\\
\text{corr}_{\vec{r}}(\vec{s}_1)[-2] &= (1)(1)+(0.2)(1)+(0.2)(-1) = 1\\
\text{corr}_{\vec{r}}(\vec{s}_1)[-1] &= (1)(1)+(1)(1)+(0.2)(-1)+(0.2)(1) = 2\\
\text{corr}_{\vec{r}}(\vec{s}_1)[0] &= (-1.2)(1)+(1)(1)+(1)(-1)+(0.2)(1)+(0.2)(1) = -0.8\\
\text{corr}_{\vec{r}}(\vec{s}_1)[1] &= (1)(1)+(-1.2)(1)+(1)(-1)+(1)(1)+(0.2)(1) = 0\\
\text{corr}_{\vec{r}}(\vec{s}_1)[2] &= (1)(1)+(1)(1)+(-1.2)(-1)+(1)(1)+(1)(1) = \mathbf{5.2}\\
\text{corr}_{\vec{r}}(\vec{s}_1)[3] &= (0.2)(1)+(1)(1)+(1)(-1)+(-1.2)(1)+(1)(1) = 0\\
\text{corr}_{\vec{r}}(\vec{s}_1)[4] &= (-0.2)(1)+(0.2)(1)+(1)(-1)+(1)(1)+(-1.2)(1) = -1.2\\
\text{corr}_{\vec{r}}(\vec{s}_1)[5] &= (-0.2)(1)+(0.2)(-1)+(1)(1)+(1)(1) = 1.6\\
\text{corr}_{\vec{r}}(\vec{s}_1)[6] &= (-0.2)(-1)+(0.2)(1)+(1)(1) = 1.4\\
\text{corr}_{\vec{r}}(\vec{s}_1)[7] &= (-0.2)(1)+(0.2)(1) = 0\\
\text{corr}_{\vec{r}}(\vec{s}_1)[8] &= (-0.2)(1) = -0.2
\end{aligned}
$$

$$
\begin{aligned}
\text{corr}_{\vec{r}}(\vec{s}_2)[-4] &= (0.2)(-1) = -0.2\\
\text{corr}_{\vec{r}}(\vec{s}_2)[-3] &= (0.2)(-1)+(0.2)(-1) = -0.4\\
\text{corr}_{\vec{r}}(\vec{s}_2)[-2] &= (1)(-1)+(0.2)(-1)+(0.2)(1) = -1\\
\text{corr}_{\vec{r}}(\vec{s}_2)[-1] &= (1)(-1)+(1)(-1)+(0.2)(1)+(0.2)(1) = -1.6\\
\text{corr}_{\vec{r}}(\vec{s}_2)[0] &= (-1.2)(-1)+(1)(-1)+(1)(1)+(0.2)(1)+(0.2)(1) = 1.6\\
\text{corr}_{\vec{r}}(\vec{s}_2)[1] &= (1)(-1)+(-1.2)(-1)+(1)(1)+(1)(1)+(0.2)(1) = 2.4\\
\text{corr}_{\vec{r}}(\vec{s}_2)[2] &= (1)(-1)+(1)(-1)+(-1.2)(1)+(1)(1)+(1)(1) = -1.2\\
\text{corr}_{\vec{r}}(\vec{s}_2)[3] &= (0.2)(-1)+(1)(-1)+(1)(1)+(-1.2)(1)+(1)(1) = -0.4\\
\text{corr}_{\vec{r}}(\vec{s}_2)[4] &= (-0.2)(-1)+(0.2)(-1)+(1)(1)+(1)(1)+(-1.2)(1) = 0.8\\
\text{corr}_{\vec{r}}(\vec{s}_2)[5] &= (-0.2)(-1)+(0.2)(1)+(1)(1)+(1)(1) = 2.4\\
\text{corr}_{\vec{r}}(\vec{s}_2)[6] &= (-0.2)(1)+(0.2)(1)+(1)(1) = 1\\
\text{corr}_{\vec{r}}(\vec{s}_2)[7] &= (-0.2)(1)+(0.2)(1) = 0\\
\text{corr}_{\vec{r}}(\vec{s}_2)[8] &= (-0.2)(1) = -0.2
\end{aligned}
$$

The maximum correlation value is 5.2 at $k = 2$. Since we have a plus-minus 1 signal of length 5, this high correlation likely comes from the satellite 1 transmission.

(b) Now your cellphone receives a new signal $r[n]$ as below. What the satellites that are transmitting and what is the delay between each satellite and your cellphone?



Answer: We want to find shifts k_1 and k_2 such that: $\vec{r}[n] = \vec{s}_1[n-k_1] + \vec{s}_2[n-k_2]$.
We calculate both $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ and $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ for different shifts k. The index where the maximum correlation value is achieved will tell us the shift indices (delays).

$$\text{corr}_{\vec{r}}(\vec{s}_1)[-3] = (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[-2] = (1)(1) + (1)(1) = 2$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[-1] = (-1)(1) + (1)(1) + (1)(-1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[0] = (2)(1) + (-1)(1) + (1)(-1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[1] = (2)(1) + (2)(1) + (-1)(-1) + (1)(1) + (1)(1) = \mathbf{7}$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[2] = (1)(1) + (2)(1) + (2)(-1) + (-1)(1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[3] = (-1)(1) + (1)(1) + (2)(-1) + (2)(1) + (-1)(1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[4] = (-1)(1) + (-1)(1) + (1)(-1) + (2)(1) + (2)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[5] = (-1)(1) + (-1)(-1) + (1)(1) + (2)(1) = 3$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[6] = (-1)(-1) + (-1)(1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[7] = (-1)(1) + (-1)(1) = -2$$
$$\text{corr}_{\vec{r}}(\vec{s}_1)[8] = (-1)(1) = -1$$

$$\text{corr}_{\vec{r}}(\vec{s}_2)[-3] = (1)(-1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[-2] = (1)(-1) + (1)(-1) = -2$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[-1] = (-1)(-1) + (1)(-1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[0] = (2)(-1) + (-1)(-1) + (1)(1) + (1)(1) = 1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[1] = (2)(-1) + (2)(-1) + (-1)(1) + (1)(1) + (1)(1) = -3$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[2] = (1)(-1) + (2)(-1) + (2)(1) + (-1)(1) + (1)(1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[3] = (-1)(-1) + (1)(-1) + (2)(1) + (2)(1) + (-1)(1) = 3$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[4] = (-1)(-1) + (-1)(-1) + (1)(1) + (2)(1) + (2)(1) = \mathbf{7}$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[5] = (-1)(-1) + (-1)(1) + (1)(1) + (2)(1) = 3$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[6] = (-1)(1) + (-1)(1) + (1)(1) = -1$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[7] = (-1)(1) + (-1)(1) = -2$$
$$\text{corr}_{\vec{r}}(\vec{s}_2)[8] = (-1)(1) = -1$$

The maximum correlation between signals $\vec{r}$ and $\vec{s}_1$ was achieved at $k_1 = 1$, and the maximum correlation between signals $\vec{r}$ and $\vec{s}_2$ was achieved at $k_2 = 4$.

