EECS 16A Fall 2019 — Designing Information Devices and Systems I

# Discussion 2A

**Definition:** A **linear combination** (also sometime referred to as a weighted combination) of vectors, $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n$, is given as a weighted summation of the vectors,

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n, \tag{1}$$

where weights, $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, scale the vectors.

## 1. Finding The Bright Cave

Nara the one-handed druid and Kody the one-handed ranger find themselves in dire straits. Before them is a cliff with four cave entrances arranged in a square: two upper caves and two lower caves. Each entrance emits a certain amount of light, and the two wish to find exactly the amount of light coming from each cave. Here's the catch: after contracting a particularly potent strain of ghoul fever, our intrepid heroes are only able to see the total intensity of light before them (so their eyes operate like a single-pixel camera). Kody and Nara are capable adventurers, but they don't know any linear algebra – and they need your help.

Kody proposes an imaging strategy where he uses his hand to completely block the light from two caves at a time. He is able to take measurements using the following four masks (black means the light is blocked from that cave):

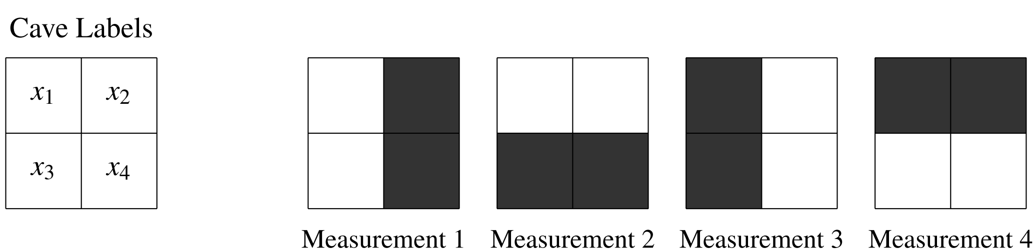



Figure 1: Four image masks.

(a) Let $\vec{x}$ be the four-element vector that represents the magnitude of light emanating from the four cave entrances. Write a matrix $\mathbf{K}$ that performs the masking process in Figure 1 on the vector $\vec{x}$, such that $\mathbf{K}\vec{x}$ is the result of the four measurements.

**Answer:**

$$\vec{m} = \mathbf{K}\vec{x}$$

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Note here that $\vec{m}$ is the vector of Kody's measurements. The order of the rows does not matter (as long as you tell us which measurement they each correspond to), but the order of the columns does. Re-arranging the columns results in a different set of masks.

(b) Does Kody's set of masks give us a unique solution for all four caves' light intensities? Why or why not?

**Answer:**

There are two ways to arrive at the answer. We will show both.

i. We can perform Gaussian elimination on the matrix. Now, since we don't know Kody's measurements (the vector $\vec{m}$), we will not augment the matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \tag{2}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_1 - m_2 + m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_1 - m_2 + m_3 \\ 0 & 0 & 0 & 0 & m_4 - m_1 + m_2 - m_3 \end{array}\right] \tag{3}$$

The matrix above has a row of zeroes, which implies that there will either be infinite solutions or no solutions. Therefore, Kody's set of masks cannot give us a unique solution for all four caves' light intensities.

ii. The second way we can show that we will not get a unique solution is to notice the equations. If we find that we could get one equation from the other equations, then we know that the solution is not unique. Notice that the sum of the first and the third row is the same is the sum of the second and fourth row.

$$\begin{aligned} m_1 + m_3 &= m_2 + m_4 \\ m_4 &= m_1 + m_3 - m_2 \\ (x_3 + x_4) &= (x_1 + x_3) + (x_2 + x_4) - (x_1 + x_2) \\ x_3 + x_4 &= x_3 + x_4 \end{aligned}$$

(c) Nara, in her infinite wisdom, places her one hand diagonally across the entrances, covering two of the cave entrances. However, her hand is not wide enough, letting in 50% of the light from the caves covered and 100% of the light from the caves not covered. The following diagram shows the percentage of light let through from each cave:

| | |
|---|---|
| 50% | 100% |
| 100% | 50% |

Does this additional measurement give them enough information to solve the problem? Why or why not?

**Answer:**

The answer is yes; the additional measurement does give them enough information to solve the problem. Since Nara's measurement is linearly independent from the other four, we are now able to solve for all four light intensities uniquely.

This can be shown using Gaussian elimination with the addition of the following equation:

$$m_5 = \frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4$$

At this point you can either add this equation to make a $5 \times 4$ system of equations, or you can remove one of Kody's masks to make a $4 \times 4$ system of equations. Here, we write it as a $5 \times 4$ matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \\ 0.5 & 1 & 1 & 0.5 & m_5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \\ 0 & 1 & 0.5 & 0.5 & m_5 - \frac{m_1}{2} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 1 & 1 & m_4 \\ 0 & 0 & 1.5 & 0.5 & m_5 + \frac{m_1}{2} - m_2 \end{array}\right] \tag{4}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \\ 0 & 0 & 0 & -1 & m_5 - \frac{3m_3}{2} + \frac{m_2}{2} - m_1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & -1 & m_5 - \frac{3m_3}{2} + \frac{m_2}{2} - m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right] \tag{5}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & 1 & -m_5 + \frac{3m_3}{2} - \frac{m_2}{2} + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 0 & m_5 - \frac{m_3}{2} - \frac{m_2}{2} \\ 0 & 0 & 0 & 1 & -m_5 + \frac{3m_3}{2} - \frac{m_2}{2} + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right] \tag{6}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -m_5 + \frac{m_3}{2} + \frac{m_2}{2} + m_1 \\ 0 & 1 & 0 & 0 & m_5 - \frac{m_3}{2} + \frac{m_2}{2} - m_1 \\ 0 & 0 & 1 & 0 & m_5 - \frac{m_3}{2} - \frac{m_2}{2} \\ 0 & 0 & 0 & 1 & -m_5 + \frac{3m_3}{2} - \frac{m_2}{2} + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right] \tag{7}$$

Notice here that, despite of the row of zeros, we still have four pivot columns. In other words, we have a system of four unkowns and four linearly independent equations. Therefore, we can uniquely determine all four light intensities given Nara's added measurement. Also notice here that the measurements do not determine how we perform our Gaussian elimination.

**2. How many solutions?**

(a) We are given a system of equations as the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 6 & 4 & 10 \\ 1 & -3 & 3 & 13 \\ 0 & 0 & 3 & 12 \end{array}\right] \tag{8}$$

Use Gaussian elimination to determine how many solutions the system of equations has.

i. Unique solution
ii. Infinite solutions
iii. No solutions

**Answer:**
Unique solution!!! Start by using Gaussian elimination to reduce the rows:

$$\left[\begin{array}{ccc|c} 2 & 6 & 4 & 10 \\ 1 & -3 & 3 & 13 \\ 0 & 0 & 3 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 1 & -3 & 3 & 13 \\ 0 & 0 & 3 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & -6 & 1 & 8 \\ 0 & 0 & 3 & 12 \end{array}\right] \tag{9}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & \frac{-1}{6} & \frac{-8}{6} \\ 0 & 0 & 3 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & \frac{-1}{6} & \frac{-8}{6} \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & -3 \\ 0 & 1 & 0 & \frac{-4}{6} \\ 0 & 0 & 1 & 4 \end{array}\right] \tag{10}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{-4}{6} \\ 0 & 0 & 1 & 4 \end{array}\right] \tag{11}$$

In this form, we can read off the single unique solution: $x = -1, y = \frac{-4}{6}, z = 4$.

(b) We are given a system of equations as the augmented matrix: $\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right]$. Use Gaussian elimination to determine how many solutions the system of equations has.

i. Unique solution
ii. Infinite solutions
iii. No solutions

**Answer:**
Infinite! Start by using Gaussian elimination to reduce the rows:

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & \frac{-1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & \frac{-1}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \tag{12}$$

The second equation determines that the third unknown, $z$ is $\frac{1}{2}$ and then there are an infinite combination of the first and second unknowns that can satisfy the first equation. To describe this set of solutions, let the second unknown, $y$, be a free variable, $t \in \mathbb{R}$, and solve for the first unknown, $x$, in terms of $y$. Exactly,

$$x = \frac{t}{3}, \quad y = t, \quad z = \frac{1}{2}. \tag{13}$$

(c) We are given the system of equations:

$$\begin{bmatrix} 2x & + & 4y & + & 2z & = & 8 \\ x & + & y & + & z & = & 6 \\ x & - & y & - & z & = & 4 \end{bmatrix} \tag{14}$$

Use Gaussian elimination to determine how many solutions the system of equations has.

i. Unique solution
ii. Infinite solutions
iii. No solutions

**Answer:** There is single unique solution. The steps to arrive at this solution can be found in Note 1 Example 1.6 (`https://inst.eecs.berkeley.edu/~ee16a/fa18/lectures/Note1.pdf`)

(d) We are given the system of equations:

$$\begin{bmatrix} x & + & y & + & 2z & = & 2 \\ & & y & + & z & = & 0 \\ 2x & + & y & + & 3z & = & 4 \end{bmatrix} \tag{15}$$

Use Gaussian elimination to determine how many solutions the system of equations has.

i. Unique solution
ii. Infinite solutions
iii. No solutions

**Answer:** There are infinite solutions! The steps to arrive at this solution can be found in Note 1 Example 1.7 (`https://inst.eecs.berkeley.edu/~ee16a/fa18/lectures/Note1.pdf`)

(e) True or False: A system of equations with more equations than unknowns will always have either infinite solutions or no solutions.

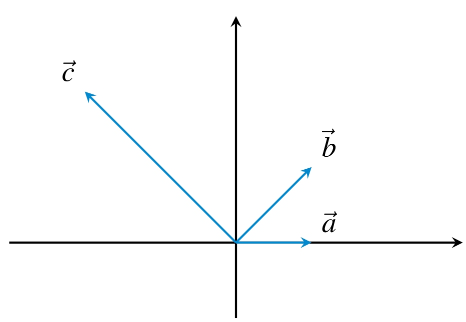
**Answer:** False, a counter example of this is when we have $N$ equations and $K$ unknowns ($N > K$) and $N-K$ of the equations are linear combinations of the first $K$. This means that there are actually $K$ unique equations and $K$ unique unknowns; therefore a unqiue solution will exist. This can be observed when Gaussian elimination is performed and the last $N-K$ rows are all 0.
For example, here is a system of four equations and two unknowns,

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 4 & 2 \\ 2 & 5 & 3 \end{array}\right] \xrightarrow{-2R_1+R_3\to R_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 5 & 3 \end{array}\right] \xrightarrow{-2R_1-R_2+R_3\to R_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{16}$$

Solving, we get a single exact solution, $x = -1$ and $y = 1$.

**3. Visualizing Span**

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount $\alpha$, then change direction, and travel along $\vec{b}$ for some amount $\beta$. We want to find these two scalars $\alpha$ and $\beta$, such that we reach point $\vec{c}$. That is, $\alpha\vec{a} + \beta\vec{b} = \vec{c}$.



(a) First, consider the case where $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{c} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Draw these vectors on a sheet of paper. Now find the two scalars $\alpha$ and $\beta$, such that we reach point $\vec{c}$. What are these scalars if we use $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ instead?

**Answer:** First set: $\alpha = -4, \beta = 2$
Second set: $\alpha = 6, \beta = -4$

(b) Formulate the system of equations as a matrix to find the unknowns, $\alpha, \beta$, in terms of the vectors $\vec{a}, \vec{b}, \vec{c}$.

**Answer:**

$$\begin{cases} \alpha a_1 + \beta b_1 = c_1 \\ \alpha a_2 + \beta b_2 = c_2 \end{cases}$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \vec{c} \tag{17}$$