EECS 16A Designing Information Devices and Systems I
Fall 2019 Discussion 3A

1. Matrix Multiplication

Consider the following matrices:

$$\mathbf{A_1} = \begin{bmatrix} 1 & 4 \end{bmatrix} \quad \mathbf{B_1} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

For each matrix multiplication problem, if the product exists, find the product by hand. Otherwise, explain why the product does not exist.

(a) $\mathbf{A_1 B_1}$

Answer: A_1 is a 1×2 vector and B_1 is a 2×1 vector, so the product exists. $\mathbf{A_1 B_1} = 11$.

(b) $\mathbf{AB}$

Answer: Since both $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices, the product exists and is a 2×2 matrix.

$$\mathbf{AB} = \begin{bmatrix} 11 & 6 \\ 12 & 7 \end{bmatrix}.$$

(c) $\mathbf{BA}$

Answer: Since both $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices, the product exists and is a 2×2 matrix.

$$\mathbf{BA} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}.$$

(d) $\mathbf{AC}$

Answer: Since $\mathbf{A}$ is a 2×2 matrix and $\mathbf{C}$ is a 2×4 matrix, the product exists and is a 2×4 matrix.

$$\mathbf{AC} = \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}.$$

(e) $\mathbf{DC}$

Answer: Since $\mathbf{C}$ is a 2×4 matrix and $\mathbf{D}$ is a 4×3 matrix, the product does not exist. This is because the number of columns in the first matrix ($\mathbf{D}$) should match the number of rows in the second matrix ($\mathbf{C}$) for this product to be defined.

(f) $\mathbf{CD}$ (Write down the dimensions of the product if it exists. For practice, you can compute the product on your own)

Answer: Since $\mathbf{C}$ is a 2×4 matrix and $\mathbf{D}$ is a 4×3 matrix, the product exists and is a 2×3 matrix.

$$\mathbf{CD} = \begin{bmatrix} 100 & 33 & 75 \\ 52 & 29 & 56 \end{bmatrix}.$$

(g) $\mathbf{EF}$ (Practice on your own)

Answer: Since $\mathbf{E}$ and $\mathbf{F}$ are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{EF} = \begin{bmatrix} 53 & 50 & 64 \\ 34 & 70 & 57 \\ 33 & 90 & 44 \end{bmatrix}.$$

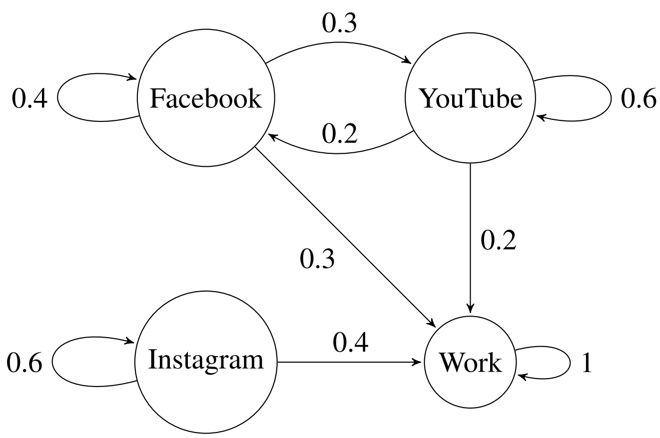
(h) **FE** (Practice on your own)

Answer: Since **E** and **F** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{FE} = \begin{bmatrix} 65 & 56 & 59 \\ 40 & 59 & 66 \\ 45 & 62 & 43 \end{bmatrix}.$$

2. Social Media

As a tech-savvy Berkeley student, the distractions of social media are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Facebook or YouTube? How do other students manage to get stuff done and balance pursuing Insta-fame? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if 100 students are on Facebook, in the next timestep, 30 of them will click on a link and move to YouTube.



(a) Derive the corresponding transition matrix.

Answer:

Let us define $x_F(t)$ as the number of students on Facebook at time t, $x_Y(t)$ as the number of students on YouTube at time t, $x_I(t)$ as the number of students on Instagram at time t, and $x_W(t)$ as the number of students working at time t. Let us now explicitly write the equations that we can then use to determine the state transition matrix.

$x_F(t+1) = 0.4x_F(t) + 0.2x_Y(t)$
$x_Y(t+1) = 0.3x_F(t) + 0.6x_Y(t)$
$x_I(t+1) = 0.6x_I(t)$
$x_W(t+1) = 0.3x_F(t) + 0.2x_Y(t) + 0.4x_I(t) + x_W(t)$

Let $\vec{x}(t) = \begin{bmatrix} x_F(t) \\ x_Y(t) \\ x_I(t) \\ x_W(t) \end{bmatrix}$.

We can now solve for the state transition matrix A such that:

$\vec{x}(t+1) = A\vec{x}(t)$.

A is therefore equal to:

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}$$

(b) There are 1500 of you in the class. Suppose on a given Friday evening (the day when HW is due), there are 700 EECS16A students on Facebook, 450 on YouTube, 200 on Instagram, and 150 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

Answer:

$$\begin{bmatrix} 370 \\ 480 \\ 120 \\ 530 \end{bmatrix}$$

(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

Answer:

Since each column's sum is equal to 1, the system is conservative. This means that we aren't losing students after each time step.

(d) You want to predict how many students will be on each website n timesteps in the future. How would you formulate that mathematically? Without working it out, can you predict roughly how many students will be in each state 100 timesteps in the future?

Answer:

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}^n \vec{x}[0] = \vec{x}[n]$$

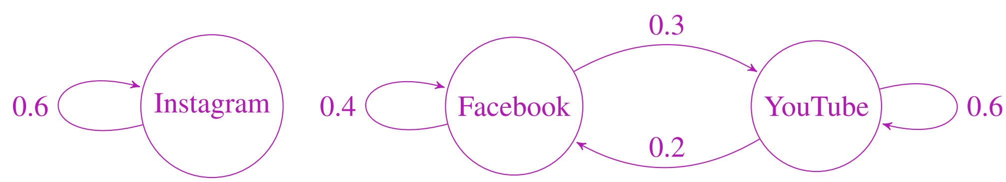
All of them will be working! Yay! With this particular system, 'Work' is called a 'final accepting state' or an 'absorbing state.' This means all the students, after jumping around and being distracted for some amount of time, will eventually end up working. Why is this? 'Work' is the only state where 100% of students who are working remain working. So as time passes, a student has some probability of landing in Work but 0 probability of leaving Work. If you actually calculate $\mathbf{A}^{100}$, you'll see that all the "mass" in the problem transfers to the bottom row, numerically reflecting the fact that 'Work' is absorbing all of the students.

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}^{100} = \begin{bmatrix} 6.83599885 \cdot 10^{-13} & 8.30745059 \cdot 10^{-13} & 0 & 0 \\ 1.24611759 \cdot 10^{-12} & 1.51434494 \cdot 10^{-12} & 0 & 0 \\ 0 & 0 & 6.53318624 \cdot 10^{-23} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The above was calculated using IPython notebook.

(e) **Challenging Practice Problem:** Suppose, instead of having 'Work' as an explicit state, we assume that any student not on Facebook/Youtube/Instagram is working. Work is like the "void," and if a student is "leaked" from any of the other states, we assume s/he has gone to work and will never come back. How would you reformulate this problem? Redraw the figure and rewrite the appropriate transition matrix. What are the major differences between this problem and the previous one?

Answer:



$$\begin{bmatrix} 0.4 & 0.2 & 0 \\ 0.3 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}$$

Since we don't track students who have gone to work, the entries in the columns of the state transition matrix no longer sum to 1. Because they sum to a number less than 1, the system is not conservative and eventually all students will disappear from the system.

$$\begin{bmatrix} 0.4 & 0.2 & 0 \\ 0.3 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}^{100} = \begin{bmatrix} 0.0684 \cdot 10^{-11} & 0.0831 \cdot 10^{-11} & 0 \\ 0.1246 \cdot 10^{-11} & 0.1514 \cdot 10^{-11} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

(a)
$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

(b) **(Practice)**
$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$$

In other words, we can swap the order of our spanning vectors and not change their span.

Answer:

(a) Suppose we have some arbitrary $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \left(\frac{a_1}{\alpha}\right)\alpha\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n.$$

Scalar multiplication cancels out. Thus, we have shown that $\vec{q} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars b_i:

$$\vec{r} = b_1(\alpha\vec{v}_1) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = (b_1\alpha)\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we now have $\text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.

(b) Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_2\vec{v}_2 + a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

Swapping the order in addition does not affect the sum, so $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$. Similarly, starting with some $\vec{r} \in \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$, again swapping the order does not affect the sum, so putting both together, the spans are thus the same.