EECS 16A Designing Information Devices and Systems I
Fall 2019 Discussion 3B

1. Mechanical Inverses

In each part, determine whether the inverse of $\mathbf{A}$ exists. If it exists, find it.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer:

We use Gaussian elimination (also known as the Gauss-Jordan method):

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \overset{R_2 \leftarrow \frac{1}{9}R_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$.

(b) $\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$

Answer:

We use Gaussian elimination:

$$\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \overset{R_1 \leftarrow R_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 5 & 4 & 1 & 0 \end{array}\right]$$

$$\overset{R_2 \leftarrow -5R_1 + R_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -5 \end{array}\right] \overset{R_1 \leftarrow R_1 + R_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & -1 & 1 & -5 \end{array}\right]$$

$$\overset{R_2 \leftarrow -R_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 5 \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$.

(c) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

Answer:

We use Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftarrow \frac{1}{5}R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_2 \leftarrow \frac{1}{2}R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \overset{R_2 \leftarrow R_2 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \qquad \overset{R_3 \leftarrow R_3 + R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & \frac{1}{2} & 1 \end{array}\right].$$

While row-reducing, we notice that the second column doesn't have a pivot (and that there is also a row of zeros). Therefore, no inverse exists.

(d) (PRACTICE)

$$\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 0 & 4 \end{bmatrix}$$

Answer:

We use Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \qquad \overset{R_1 \leftarrow \frac{1}{5} R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_2 \leftarrow \frac{1}{2} R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \qquad \overset{R_2 \leftarrow R_2 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \qquad \overset{R_2 \leftrightarrow R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right]$$

$$\overset{R_2 \leftarrow -R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right] \qquad \overset{R_3 \leftarrow -R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\overset{R_2 \leftarrow R_2 + R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right] \qquad \overset{R_1 \leftarrow R_1 - 3R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{2}{5} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

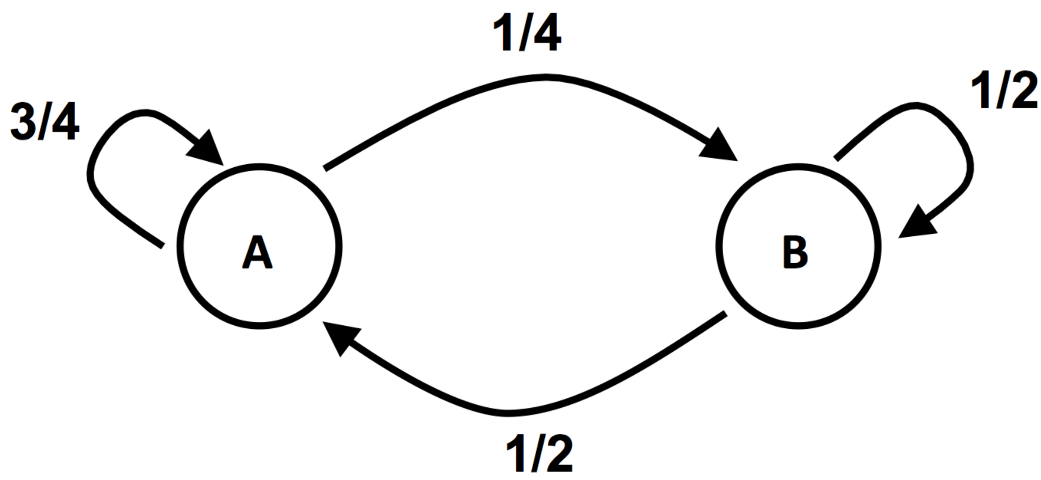
$$\overset{R_1 \leftarrow R_1 - R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{5} & 2 & 1 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{4}{5} & 2 & 1 \\ \frac{2}{5} & -\frac{1}{2} & -1 \\ \frac{1}{5} & -\frac{1}{2} & 0 \end{bmatrix}$.

2. Transition Matrix

(a) Suppose there exists some network of pumps as shown in the diagram below. Let $\vec{x}(n) = \begin{bmatrix} x_A(n) \\ x_B(n) \end{bmatrix}$ where $x_A(n)$ and $x_B(n)$ are the states at timestep n.

Find the state transition matrix S, such that $\vec{x}(n+1) = S\vec{x}(n)$.



Answer:
We can write the following equations by examining the state transition diagram:
$x_A(n+1) = (3/4)x_A(n) + (1/2)x_B(n)$

$x_B(n+1) = (1/4)x_A(n) + (1/2)x_B(n).$

From here, we can directly write down the state transition matrix as $S = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$.

(b) Let us now find the matrix S^{-1} such that we can recover $\vec{x}(n-1)$ from $\vec{x}(n)$. Specifically, solve for S^{-1} such that $\vec{x}(n-1) = S^{-1}\vec{x}(n)$.

Answer:
We can use Gaussian elimination to solve for the matrix S^{-1}, i.e. inverse of the matrix S that we just found:

$$\left[\begin{array}{cc|cc} 3/4 & 1/2 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow \frac{4}{3}R_1} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow -4R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ -1 & -2 & 0 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ -1 & -2 & 0 & -4 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -4/3 & 4/3 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -4/3 & 4/3 & -4 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow \frac{-3}{2} R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 3 \end{array}\right].$$

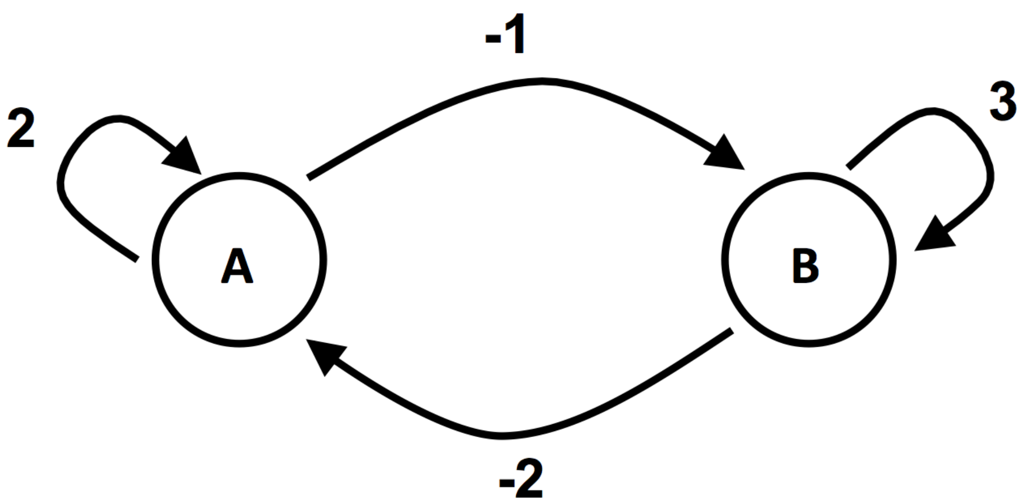
Therefore:

$$S^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}.$$

Note that the columns of S^{-1} still sum to 1.

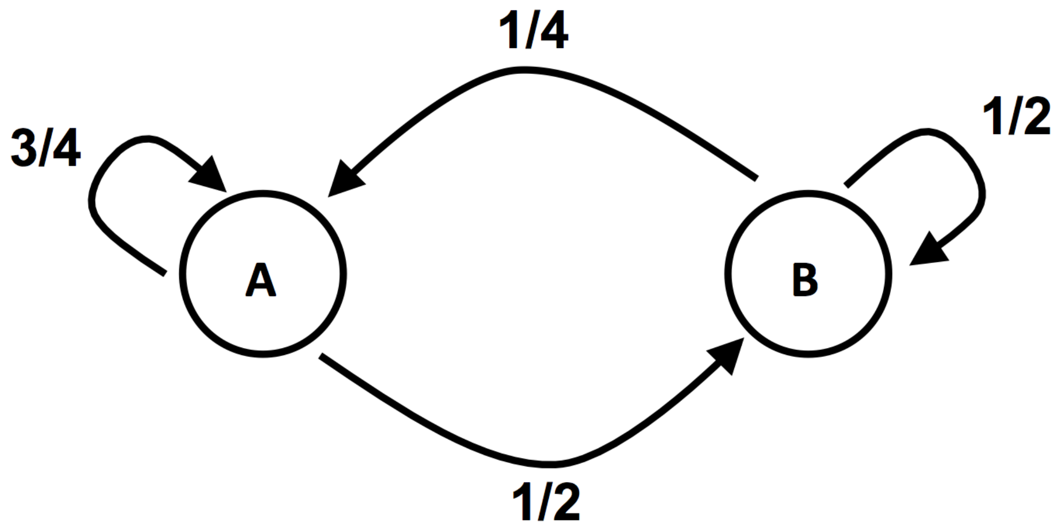
(c) Now draw the state transition diagram that corresponds to the S^{-1} that you just found.

Answer:



The matrix S^{-1} can be thought of as the matrix that *turns back time* for the pump system. Although it is non-physical, the weights that have an absolute value greater than 1 can be thought of as "generating" water, and the weights that have negative weight can be thought of as "destroying" water. However, note that the outflow weights of each node still sum to 1 (i.e. the columns of S^{-1} still sum to 1). This means that in total all of the water is being conserved during the transition between time steps, even when time is reversed.

(d) Redraw the diagram from the first part of the problem, but now with the directions of the arrows reversed. Let us call the state transmission matrix of this "reversed" state transition diagram T. Does $T = S^{-1}$?



Answer:

After drawing the "reversed" state transition diagram, we can write the following equations:

$x_A(n+1) = (3/4)x_A(n) + (1/4)x_B(n)$

$x_B(n+1) = (1/2)x_A(n) + (1/2)x_B(n)$.

From here, we can directly write down the state transition matrix as: $T = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$.

Note that $T \neq S^{-1}$. What we have actually found is that T is equal to the *transpose* of S, denoted by S^T (the superscript T denotes the transpose of a matrix). The transpose of a matrix is when its rows become its columns. In general, a matrix's inverse and its transpose are not equal to each other.