EECS 16A Designing Information Devices and Systems I
Fall 2019 Discussion 4A

1. Proofs

(a) Prove the following statement (proved earlier in lecture): If the columns of A are linearly dependent, then $A\vec{x} = \vec{b}$ does not have a unique solution.

Answer: Let's walk through this proof step by step: we'll start by assuming we have a matrix A with linearly dependent columns, and then we will show that this means that the system does not have a unique solution.

Since we are interested in the columns of A, let's start by explicitly defining the columns of A:

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix},$$

We've defined A to have linearly dependent columns, so by the definition of linear dependence, there exist scalars $\alpha_1, \dots, \alpha_n$ such that $\alpha_1\vec{a}_1 + \dots + \alpha_n\vec{a}_n = \vec{0}$ where not all of the α_i's are zero. We can put these α_i's in a vector

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

and by the definition of matrix-vector multiplication, we can compactly write the expression above:

$$A\vec{\alpha} = \vec{0}$$

where $\vec{\alpha} \neq \vec{0}$.

Recall that we are trying to show that the system of equations $A\vec{x} = \vec{b}$ does not have a unique solution. We know that systems of equations can have either zero, one, or infinite solutions. If our system of equations has zero solutions, then it cannot have a unique solution, so we don't need to consider this case. Now let's consider the case where we have at least one solution, $\vec{x}$:

$$\begin{aligned} A\vec{x} &= \vec{b} \\ A\vec{x} + \vec{0} &= \vec{b} \\ A\vec{x} + A\vec{\alpha} &= \vec{b} \\ A(\vec{x} + \vec{\alpha}) &= b \end{aligned}$$

Therefore, $\vec{x} + \vec{\alpha}$ is also a solution to the system of equations! Since both $\vec{x}$ and $\vec{x} + \vec{\alpha}$ are solutions, and $\vec{\alpha} \neq \vec{0}$, the system has more than one solution. We've now proven the theorem.

Note that we can add any multiple of $\vec{\alpha}$ to $\vec{x}$ and it will still be a solution – therefore, if there is at least one solution to the system and the columns of A are linearly dependent, then there are infinite solutions.

(b) Often, when one is asked to prove something you are asked to prove something of the following nature:

- $P \implies Q$. This is read as P implies Q.

Identify P and Q in the theorem you just proved above.

There are a couple of things to remember when reading these statements. First, is that the direction of implication matters.

- If you prove $P \implies Q$, this does not mean that $Q \implies P$ is also true.

Suppose someone tells you that $P \implies Q$ is true. Then you find out later that Q is actually false. What can you say about P?

- If $P \implies Q$ and Q is false, then P must be false.

Answer: In the theorem above

- P = The columns of A are linearly dependent.
- Q = $A\vec{x} = \vec{b}$ does not have a unique solution.

Consider the simple example:

- P = It is raining.
- Q = There are clouds.
- $P \implies Q$ should be read literally as: If it is raining, then there are clouds.
- Note 1. This **does not** mean: If there are clouds, it is raining. (There could be clouds without rain).
- Note 2. This **does** however mean: If there are no clouds, it is not raining. (Because rain requires there to be clouds.)

2. Identifying a Basis

Does each of these sets of vectors describe a basis for $\mathbb{R}^3$? If the vectors do not form a basis for $\mathbb{R}^3$, can they be thought of as a basis for some other vector space?

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Answer:

- V_1: The vectors are linearly independent, but they are not a basis for $\mathbb{R}^3$, because you cannot construct all vectors in $\mathbb{R}^3$ using these vectors. Instead, they are a basis for some 2-dimensional subspace of $\mathbb{R}^3$. This subspace can be described by $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.
- V_2: Yes, the vectors are linearly independent and will form a basis for $\mathbb{R}^3$. To check that the vectors are linearly independent, you should do Gaussian Elimination of the matrix of the columns: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Check that you can get all the way to identity, i.e. the system has a unique solution.
- V_3: No, $\vec{v}_2 + \vec{v}_3 = \vec{v}_1$, so the vectors are linearly dependent. Hence, they cannot form a basis for any vector space of any dimension.

3. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $\mathbf{A}$? What is its dimension?

ii. What is the null space of $\mathbf{A}$? What is its dimension?

iii. Are the column spaces of the row reduced matrix $\mathbf{A}$ and the original matrix $\mathbf{A}$ the same?

iv. Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Answer:

Column space: $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

Null space: $\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.

Not a basis for $\mathbb{R}^2$.

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Answer:

Column space: $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Null space: $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

The two column spaces are not the same.

Not a basis for $\mathbb{R}^2$.

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Answer:

Column space: $\mathbb{R}^2$

Null space: $\text{span}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

The two column spaces are the same as the column span $\mathbb{R}^2$.

This is a basis for $\mathbb{R}^2$.

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

Answer:

Column space: $\text{span}\left\{\begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix}\right\}$

Null space: $\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

The two column spaces are not the same.

Not a basis for $\mathbb{R}^2$.

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

Answer:

i. The columspace of the columns is $\mathbb{R}^2$. The columns of $\mathbf{A}$ do not form a basis for $\mathbb{R}^2$. This is because the columns of $\mathbf{A}$ are linearly dependent.

ii. The following algorithm can be used to solve for the null space of a matrix. The procedure is essentially solving the matrix-vector equation $\mathbf{A}\vec{x} = \vec{0}$ by performing Gaussian elimination on $\mathbf{A}$. We start by performing Gaussian elimination on matrix $\mathbf{A}$ to get the matrix into upper-triangular form.

$$\begin{aligned} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \text{reduced row echelon form} \end{aligned}$$

$$\begin{aligned} &x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0 \\ &x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0 \\ &x_3 \text{ is free and } x_4 \text{ is free} \end{aligned}$$

Now let $x_3 = s$ and $x_4 = t$. Then we have:

$$\begin{aligned} x_1 + \frac{1}{2}s - \frac{7}{2}t = 0 \\ x_2 + \frac{5}{2}s + \frac{1}{2}t = 0 \end{aligned}$$

Now writing all the unknowns (x_1, x_2, x_3, x_4) in terms of the dummy variables:

$$\begin{aligned} x_1 &= -\frac{1}{2}s + \frac{7}{2}t \\ x_2 &= -\frac{5}{2}s - \frac{1}{2}t \\ y &= s \\ z &= t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s + \frac{7}{2}t \\ -\frac{5}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s \\ -\frac{5}{2}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2}t \\ -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

So every vector in the nullspace of $\mathbf{A}$ can be written as follow:

$$\text{Nullspace}(\mathbf{A}) = s \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore the nullspace of $\mathbf{A}$ is

$$\text{span}\left\{ \left[\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right] \right\}$$

$\mathbf{A}$ has a 2-dimensional null space.

iii. In this case, the column space of the row reduced matrix is also $\mathbb{R}^2$, but this need not be true in general.

iv. No the columns of $\mathbf{A}$ do not form a basis for $\mathbb{R}^2$.