EECS 16A Designing Information Devices and Systems I
Fall 2019 Discussion 4B

1. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of **A**? What is its dimension?

ii. What is the null space of **A**? What is its dimension?

iii. Are the column spaces of the row reduced matrix **A** and the original matrix **A** the same?

iv. Do the columns of **A** form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.

Not a basis for $\mathbb{R}^2$.

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

The two column spaces are not the same.

Not a basis for $\mathbb{R}^2$.

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Answer:

Column space: $\mathbb{R}^2$

Null space: span $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

The two column spaces are the same as the column span $\mathbb{R}^2$.

This is a basis for $\mathbb{R}^2$.

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$.

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

Answer:

i. The columspace of the columns is $\mathbb{R}^2$. The columns of $\mathbf{A}$ do not form a basis for $\mathbb{R}^2$. This is because the columns of $\mathbf{A}$ are linearly independent.

ii. The following algorithm can be used to solve for the null space of a matrix. The procedure is essentially solving the matrix-vector equation $\mathbf{A}\vec{x} = \vec{0}$ by performing Gaussian elimination on $\mathbf{A}$. We start by performing Gaussian elimination on matrix $\mathbf{A}$ to get the matrix into upper-triangular form.

$$\begin{aligned} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \text{ reduced row echelon form} \end{aligned}$$

$$\begin{aligned} x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0 \\ x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0 \\ x_3 \text{ is free and } x_4 \text{ is free} \end{aligned}$$

Now let $x_3 = s$ and $x_4 = t$. Then we have:

$$\begin{aligned} x_1 + \frac{1}{2}s - \frac{7}{2}t = 0 \\ x_2 + \frac{5}{2}s + \frac{1}{2}t = 0 \end{aligned}$$

Now writing all the unknowns (x_1, x_2, x_3, x_4) in terms of the dummy variables:

$$\begin{aligned} x_1 &= -\frac{1}{2}s + \frac{7}{2}t \\ x_2 &= -\frac{5}{2}s - \frac{1}{2}t \\ y &= s \\ z &= t \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s + \frac{7}{2}t \\ -\frac{5}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s \\ -\frac{5}{2}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2}t \\ -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

So every vector in the nullspace of $\mathbf{A}$ can be written as follow:

$$\text{Nullspace}(\mathbf{A}) = s \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore the nullspace of $\mathbf{A}$ is

$$\text{span} \left\{ \left[\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right] \right\}$$

$\mathbf{A}$ has a 2-dimensional null space.

iii. In this case, the column space of the row reduced matrix is also $\mathbb{R}^2$, but this need not be true in general.

iv. No the columns of $\mathbf{A}$ do not form a basis for $\mathbb{R}^2$.

2. Constructing a Basis

Let's consider a subspace of $\mathbb{R}^3$ called V which has the following property: for every vector in V, the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$, $a_1 = 2(a_2 + a_3)$.

Find a basis for V. What is the dimension of V? Is this basis unique?

Answer:

Any vector $\vec{v}$ in V is going to look as follows:

$$\vec{v} = \begin{bmatrix} 2(a_2 + a_3) \\ a_2 \\ a_3 \end{bmatrix} = a_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Now, we consider the set of vectors $\mathscr{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. The vectors are linearly independent. Furthermore, from the above equation, any vector $\vec{v} \in V$ can be expressed as a linear combination of the vectors in $\mathscr{B}$ (the corresponding coefficients are a_2 and a_3). This means that $V = \text{span}\{\mathscr{B}\}$.

Therefore,

$$\mathscr{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

forms a basis for V.

$\dim(\mathscr{B}) = 2$ (there are two vectors in $\mathscr{B}$), so the dimension of V is 2.

A basis for a space is never unique. An example to convince yourself why this is true involves scaling the basis vector(s) and realizing that this new set of vectors is still a basis for the same space. For example suppose we scale the first basis vector in the set $\mathscr{B}$ by 2. We then get the new set $\mathscr{B}_2$ being:

$$\mathscr{B}_2 = \left\{ \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

which is a different set but nonetheless also a basis for the space V.

3. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \,\middle|\, \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

Answer:

Yes, V is a subspace of $\mathbb{R}^3$. We will *prove this* by using the definition of a subspace.

First of all, note that V is a subset of $\mathbb{R}^3$ – all elements in V are of the form $\begin{bmatrix} c+d \\ c \\ c+d \end{bmatrix}$, which is a 3-dimensional real vector.

Now, consider two elements $\vec{v}_1, \vec{v}_2 \in V$ and $\alpha \in \mathbb{R}$.

This means that there exists $c_1, d_1 \in \mathbb{R}$, such that $\vec{v}_1 = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Similarly, there exists $c_2, d_2 \in \mathbb{R}$, such that $\vec{v}_2 = c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Now, we can see that

$$\vec{v}_1 + \vec{v}_2 = (c_1 + c_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (d_1 + d_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $\vec{v}_1 + \vec{v}_2 \in V$.

Also,

$$\alpha \vec{v}_1 = (\alpha c_1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (\alpha d_1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so $\alpha \vec{v}_1 \in V$.

Furthermore, we observe that the zero vector is contained in V, when we set $c = 0$ and $d = 0$.

We have thus identified V as a subset of $\mathbb{R}^3$, shown both of the no escape (closure) properties (closure under vector addition and closure under scalar multiplication), as well as the existence of a zero vector, so V is a subspace of $\mathbb{R}^3$.

It's important to note that satisfying the subset property and the two forms of closure additionally implies this subspace V also satisfies the axioms of a vector space, and therefore is definitionally also a vector space.

4. (PRACTICE) Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let's consider the vector space $\mathbb{R}^k$ and a set of n vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^k$.

(a) For the first part of the problem, let $k > n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^k$? Why/why not? What conditions would we need?

Answer:

No, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ cannot form a basis for $\mathbb{R}^k$. The dimension of $\mathbb{R}^k$ is k, so you would need k linearly independent vectors to describe the vector space. Since $n < k$, this is not possible.

(b) Let $k = n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^k$? Why/why not? What conditions would we need?

Answer:

Fact: $rank(A)$ **= the number of linearly independent columns in matrix A = the number of linearly independent rows in matrix A**

Yes, this is possible. The only condition we need is that $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent. If the vectors are linearly independent, since there are k of them, we can put them into a square matrix V:

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

This matrix is square because the number of entries in the column vectors (k) is equal to the number of column vectors (n). Using the fact from above, we know that if the square matrix V has n linearly independent columns, it will also have n linearly independent rows. Therefore the matrix vector equation $V\vec{x} = \vec{b}$ will always have a unique solution for the value of $\vec{x}$ (analogously the matrix V is invertible). Thus the columns of matrix V: $(\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\})$ has in its span all possible values of the vector $\vec{b}$ and thus spans all of $\mathbb{R}^k$. Therefore we can conclude that if $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent, $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is a basis for $\mathbb{R}^k$.

(c) Now, let $k < n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^k$? What vector space could they form a basis for?

Hint: Think about whether the vectors can be linearly independent.

Answer:

No, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ cannot form a basis for $\mathbb{R}^k$. $\mathbb{R}^k$ will be spanned by k linearly independent vectors. Any additional vectors in $\mathbb{R}^k$ must already exist in the span of the previous vectors, and are therefore linearly dependent. Since $n > k$, some of the vectors have to be linearly dependent, so they cannot form a basis.

The two regimes—one where $n > k$ and one where $n < k$—give rise to two different classes of interesting problems. You might learn more about them in upper division courses!