EECS 16A Designing Information Devices and Systems I
Fall 2019 Discussion 5A

Definition: If $A\vec{x} = \lambda\vec{x}$, then $\lambda \in \mathbb{R}$ is called an eigenvalue of A. $\vec{x}$ belongs to the eigenspace of A corresponding to eigenvalue λ. All vectors $\vec{x}$ in the eigenspace are called eigenvectors corresponding to the eigenvalue λ.

1. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix **M** and the associated eigenvectors. If the inverse of **M** exists, find it.

(a) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\begin{vmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0$$

$$-\lambda(-3-\lambda)+2=0$$
$$\lambda^2+3\lambda+2=0$$
$$\lambda=-1,-2$$

$\lambda = -1$:

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x+y=0 \implies y=-x \implies \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenspace for $\lambda = -1$ is span$\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$.

$\lambda = -2$:

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$2x+y=0 \implies y=-2x \implies \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The eigenspace for $\lambda = -2$ is span$\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\}$.

For the inverse matrix we use Gaussian elimination:

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} -2 & -3 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

$$\left[\begin{array}{cc|cc} -2 & -3 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow \frac{-R_2}{2}} \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & 0 & \frac{-1}{2} \\ 0 & 1 & 1 & 0 \end{array}\right].$$

$$\left[\begin{array}{cc|cc} 1 & \frac{3}{2} & 0 & \frac{-1}{2} \\ 0 & 1 & 1 & 0 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow R_1 - \frac{3R_2}{2}} \left[\begin{array}{cc|cc} 1 & 0 & \frac{-3}{2} & \frac{-1}{2} \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Therefore, we get $\mathbf{M}^{-1} = \begin{bmatrix} \frac{-3}{2} & \frac{-1}{2} \\ 1 & 0 \end{bmatrix}$.

(b) $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)(2-\lambda) - 2 = \lambda^2 - 3\lambda = \lambda(\lambda - 3) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 0$ and $\lambda = 3$.

For the eigenvalue $\lambda = 0$:

$$(\mathbf{M} - 0\mathbf{I})\vec{x} = \mathbf{M}\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \vec{x} = \vec{0}$$

which is simply x_1 is free and $x_2 = -x_1$ or equivalently $\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 3$:

$$(\mathbf{M} - 3\mathbf{I})\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\vec{x} = \vec{0}$$

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \vec{x} = \vec{0}$$

which is simply x_1 is free and $x_2 = 2x_1$ or equivalently $\begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

Matrix $\mathbf{M}$ has linearly dependent columns, therefore the inverse $\mathbf{M}^{-1}$ does not exist.

(c) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\right) = 0$$

The determinant of a diagonal matrix is the product of the entries.

$$(1-\lambda)(9-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 1$ and $\lambda = 9$.

For the eigenvalue $\lambda = 1$:

$$(\mathbf{M} - 1\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_2 = 0$ or equivalently $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ or equivalently $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 9$:

$$(\mathbf{M} - 9\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -8 & 0 \\ 0 & 0 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = 0$ or equivalently $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ or equivalently $\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

For the inverse matrix we use Gaussian elimination:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow \frac{1}{9}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} \end{array}\right].$$

Therefore, we get $\mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$.

(d) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\begin{vmatrix} 2-\lambda & 4 \\ 4 & 8-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(8-\lambda) - 16 = 0$$
$$\lambda^2 - 10\lambda = 0 \implies \lambda(\lambda - 10) = 0$$
$$\lambda = 0, 10$$

$\lambda = 0$:

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$$

$$2x + 4y = 0 \implies y = -\frac{1}{2}x \implies \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The eigenspace for $\lambda = 0$ is span$\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}$.

$\lambda = 10$:

$$\begin{bmatrix} 2-10 & 4 \\ 4 & 8-10 \end{bmatrix} = \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$2x - y = 0 \implies y = 2x \implies \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The eigenspace for $\lambda = 10$ is span$\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$.

Matrix $\mathbf{M}$ has linearly dependent columns, therefore the inverse $\mathbf{M}^{-1}$ does not exist.

2. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

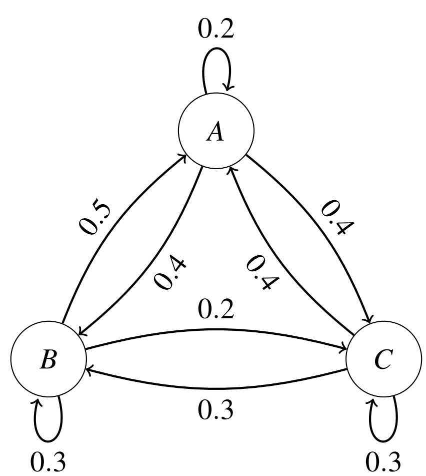

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix $\mathbf{T}$ representing the pumps system.

Answer:

$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{-\sqrt{2}-1}{10}$, $\lambda_3 = \frac{\sqrt{2}-1}{10}$ are the eigenvalues of $\mathbf{T}$. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

Answer:

We know $\lambda_1 = 1$ is the eigenvalue corresponding to the steady state eigenvector. Therefore,

$$T\vec{x} = 1\vec{x} = \lambda_1\vec{x} \Rightarrow \vec{x} \in \text{Null}\left(\mathbf{T} - 1\cdot\mathbf{I}\right) = \text{Null}\left(\begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) =$$

$$\text{Null}\left(\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix}\right)$$

In order to row reduce $\mathbf{T} - 1\cdot\mathbf{I}$ we use Gaussian elimination:

$$\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix} \overbrace{\Rightarrow}^{R_2 \leftarrow R_1 + 2R_2} \begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0 & -0.9 & 1 \\ 0.4 & 0.2 & -0.7 \end{bmatrix} \overbrace{\Rightarrow}^{R_3 \leftarrow R_1 + 2R_3} \begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0 & -0.9 & 1 \\ 0 & 0.9 & -1 \end{bmatrix}$$

$$\overbrace{\Rightarrow}^{R_3 \leftarrow R_2 + R_3} \begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0 & -0.9 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a vector describing the steady state, then we can set x_3 to be the free variable. Therefore, we can write:

$x_3 = a$, $0.9x_2 = a$ and $x_1 = \frac{0.5x_2 + 0.4x_3}{0.8} \Rightarrow x_3 = a$, $x_2 = \frac{10}{9}a$ and $x_1 = \frac{43}{36}a$

which means that our steady state vector is of the form $\begin{bmatrix} \frac{43}{36}\alpha \\ \frac{10}{9}\alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R}$.

3. Proofs

(a) Let $\mathbf{A}$ be an invertible matrix. Show that if λ is an eigenvalue of $\mathbf{A}$, then $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{A}^{-1}$.

Answer:

A is invertible,

$\Rightarrow A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$.

$\Rightarrow A\vec{x} = \vec{0}$ has a unique solution.

$\Rightarrow \vec{x} = \vec{0}$ is the only solution to $A\vec{x} = \vec{0}$

$\Rightarrow A\vec{x} = \vec{0}$ has no non zero vectors $\vec{x}$ that satisfy it.

Therefore, 0 is not an eigenvalue. Let $\vec{v}$ be the eigenvector of $\mathbf{A}$ corresponding to λ.

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

Since we know that $\mathbf{A}$ is invertible, we can left-multiply both sides by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{A}\vec{v} = \lambda\mathbf{A}^{-1}\vec{v}$$

$$\vec{v} = \lambda\mathbf{A}^{-1}\vec{v}$$

$$\mathbf{A}^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$$