EECS 16A Designing Information Devices and Systems I
Fall 2019 Discussion 5B

1. Eigenvalues and Special Matrices – Visualization

An eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where λ is a a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$.

The following parts don't require knowledge about how to find eigenvalues. Answer each part by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer:

Multiplying the identity matrix with any vector in $\mathbb{R}^n$ produces the same vector, that is, $\mathbf{I}\vec{x} = \vec{x} = 1 \cdot \vec{x}$. Therefore, $\lambda = 1$. Since $\vec{x}$ can be any vector in $\mathbb{R}^n$, the corresponding eigenvectors are all vectors in $\mathbb{R}^n$.

(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer:

Since the matrix is diagonal, multiplying the diagonal matrix with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $\mathbf{D}\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries d_i of $\mathbf{D}$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(c) Does a rotation matrix in $\mathbb{R}^2$ have any eigenvalues $\lambda \in \mathbb{R}$?

Answer:

There are three cases:

i. Rotation by 0° (more accurately, any integer multiple of 360°), which yields a rotation matrix $\mathbf{R} = \mathbf{I}$: This will have one eigenvalue of $+1$ because it doesn't affect any vector ($\mathbf{R}\vec{x} = \vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

ii. Rotation by 180° (more accurately, any angle of $180° + n \cdot 360°$ for integer n), which yields a rotation matrix $\mathbf{R} = -\mathbf{I}$: This will have one eigenvalue of -1 because it "flips" any vector ($\mathbf{R}\vec{x} = -\vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

iii. Any other rotation: there aren't any real eigenvalues. The reason is, if there were any real eigenvalue $\lambda \in \mathbb{R}$ for a non-trivial rotation matrix, it means that we can get $\mathbf{R}\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$, which means that by rotating a vector, we scaled it. This is a contradiction (again, unless $\mathbf{R} = \mathbf{I}$). Refer to Figure 1 for a visualization.

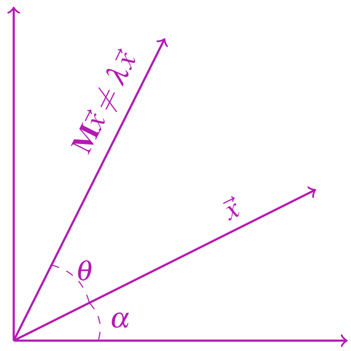

Figure 1: Rotation will never scale any non-zero vector (by a real number) unless it is rotation by an integer multiple of 360° (identity matrix) or the rotation angle is $\theta = 180° + n \cdot 360°$ for any integer n ($-\mathbf{I}$).

(d) Does a reflection matrix, where the reflection is around any line passing through the origin, have any eigenvalues $\lambda \in \mathbb{R}$?

Answer:

Yes, both $+1$ and -1. Why? Reflecting any vector that is on the reflection axis will not affect it (eigenvalue $+1$). Reflecting any vector orthogonal to the reflection axis will just "flip it/negate it" (eigenvalue -1). In other words, the axis of reflection is the eigenspace associated with the eigenvalue $+1$ and the orthogonal space to that axis of reflection is the eigenspace associated with the eigenvalue -1. Refer to Figure 2 for a visualization.

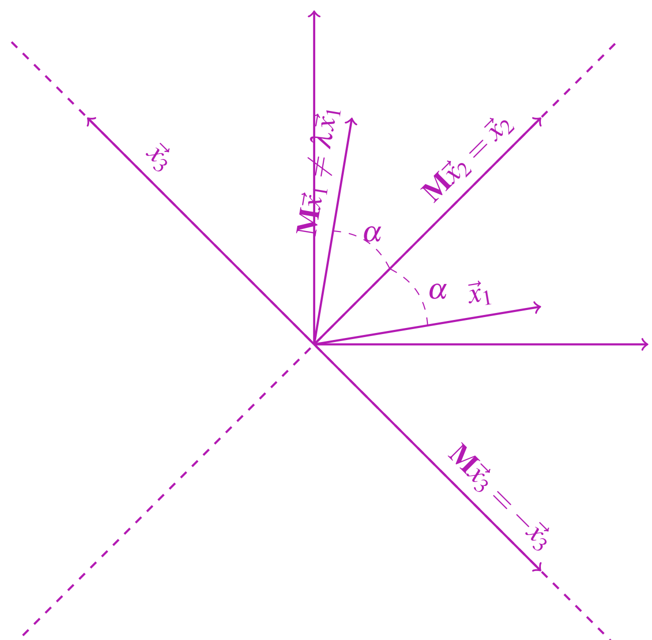

Figure 2: Reflection will scale vectors on the reflection axis (by $+1$) and orthogonal to it (by -1).

(e) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

Answer:

$\dim(\text{Null}(\mathbf{M})) > 0$
$\mathbf{M}\vec{x} = \vec{b}$ has no unique solution.

(f) Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer:

Note that the matrix has linearly dependent columns. Therefore, according to part (e), one eigenvalue is $\lambda = 0$. The corresponding eigenvector, which is equivalent to the basis vector for the null space, is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The other eigenvalue is, by inspection, $\lambda = 1$ with the corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

2. Mechanical Determinants

(a) Compute the determinant of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Answer:

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = 6$$

(b) Compute the determinant of $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

Answer:

$$\det\left(\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\right) = 6$$

3. Steady and Unsteady States

(a) You're given the matrix $\mathbf{M}$ (below) which describes some physical system (could describe either people or water):

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Find the eigenspaces associated with the following eigenvalues:

i. $\text{span}(\vec{v}_1)$, associated with $\lambda_1 = 1$
ii. $\text{span}(\vec{v}_2)$, associated with $\lambda_2 = 2$
iii. $\text{span}(\vec{v}_3)$, associated with $\lambda_3 = \frac{1}{2}$

Answer: This is practice finding the null space.

i. $\lambda = 1$:

$$\left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

ii. $\lambda = 2$

$$\left[\begin{array}{ccc|c} \frac{-3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -3 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \beta \in \mathbb{R}$$

iii. $\lambda = \frac{1}{2}$

$$\left[\begin{array}{ccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \in \mathbb{R}$$

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$. The values α, β, and γ are the coordinates for the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$
0	0	$\neq 0$		
0	$\neq 0$	0		
0	$\neq 0$	$\neq 0$		
$\neq 0$	0	0		
$\neq 0$	0	$\neq 0$		
$\neq 0$	$\neq 0$	0		
$\neq 0$	$\neq 0$	$\neq 0$		

Answer:

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	-
0	$\neq 0$	$\neq 0$	No	-
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	-
$\neq 0$	$\neq 0$	$\neq 0$	No	-