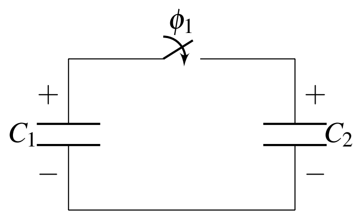
EECS 16A Designing Information Devices and Systems I
Fall 2019 Discussion 9B

1. Capacitors and Charge Conservation (with Energy!)

(a) Consider the circuit below with $C_1 = C_2 = 1\,\mu F$ and an open switch. Suppose that C_1 is initially charged to $+1\,V$ and that C_2 is charged to $+2\,V$. How much charge is on C_1 and C_2? How much energy is stored in each of the capacitors? What is the total stored energy?



Answer:

$$q_1 = C_1 V_1 = 1\,\mu C$$
$$q_2 = 2\,\mu C$$

Energy:

$$E = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV$$

Note: We include a factor of $\frac{1}{2}$ because when we charge a capacitor, we do it incrementally. Not all the charge Q is moved against a potential of V. We are basically taking $\int v(q)\,dq = \int \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C}$.

Therefore, $E_1 = 0.5\,\mu J$, $E_2 = 2\,\mu J$, and the total energy is $2.5\,\mu J$.

Note: Does it make sense that C_2 has more than twice the energy of C_1 even though it is only twice the voltage? (Yes, C_2 has *more charge* at twice the voltage.)

(b) Now the switch is closed (i.e. the capacitors are connected together.) What are the voltages across and the charges on C_1 and C_2? What is the total stored energy?

Answer:

Charge is always conserved.

Let $Q_{C_1,1}, Q_{C_2,1}$ be the charges on the capacitors after the switch is closed. There was $3\,\mu C$ of total charge on the top two plates of the capacitors initially, so we must have

$$Q_{C_1,1} + Q_{C_2,1} = 3\,\mu C$$

Further, the voltages on the capacitors must be the same, so:

$$\frac{Q_{C_1,1}}{C_1} = \frac{Q_{C_2,1}}{C_2}$$

Solving this system gives:

$$Q_{C_1,1} = Q_{C_2,1} = 1.5\,\mu C$$

Comparing to the previous part, charge has moved from C_2 to C_1. This yields a voltage of $1.5\,\text{V}$.

The energies are:

$$E_1 = \frac{1}{2}(1.5\,\mu\text{C})(1.5\,\text{V}) = 1.125\,\mu\text{J}$$

$$E_2 = \frac{1}{2}(1.5\,\mu\text{C})(1.5\,\text{V}) = 1.125\,\mu\text{J}$$

Total energy: $E = 2.25\,\mu\text{J}$

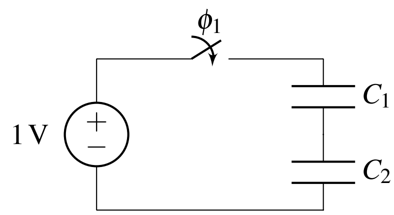
(c) Is there more or less energy than before the switch was closed? Why?

Answer:

The energy decreases when the switch is closed. First, the energy could not possibly be more since there is no energy input to the system. But there's no energy output either – where did the energy go? The wires have an internal resistance that would dissipate this energy.

(d) Answer the above three questions but now with $C_1 = 2\,\mu\text{F}$ and $C_2 = 1\,\mu\text{F}$. Suppose that they are initially charged in the same way: C_1 is charged to $+1\,\text{V}$, and C_2 is charged to $+2\,\text{V}$.

(e) Consider the following circuit with $C_1 = 1\,\mu\text{F}$ and $C_2 = 3\,\mu\text{F}$. Suppose that both capacitors are initially uncharged ($0\,\text{V}$).



What are the voltages across each capacitor after the switch is closed? What are the charges on each capacitor?

Answer:

Solution 1: Use equivalent capacitance:

$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

The series has an equivalent capacitance $C_{eq} = \frac{3}{4}\mu\text{F}$, so $Q_{eq} = C_{eq}V = \frac{3}{4}\mu\text{C}$. Note that this means $+\frac{3}{4}\mu\text{C}$ is on the top plate of C_1 (and hence the top plate of C_2, etc.).

The voltage across C_1 is $\frac{Q_{eq}}{C_1} = \frac{3}{4}\text{V}$. The voltage across C_2 is $\frac{Q_{eq}}{C_2} = \frac{1}{4}\text{V}$.

Solution 2: Let an unknown $+q$ charge be on the top plate of C_1. Then by charge conservation, $-q$ charge is on the bottom plate of C_2. And since conductors have no $\vec{E}$ field, $-q$ is on the bottom plate of C_1, and $+q$ on the top of C_2.

Now, by KVL, the voltage across the series is $1\,\text{V}$:

$$\frac{q}{C_1} + \frac{q}{C_2} = 1\,\text{V}$$

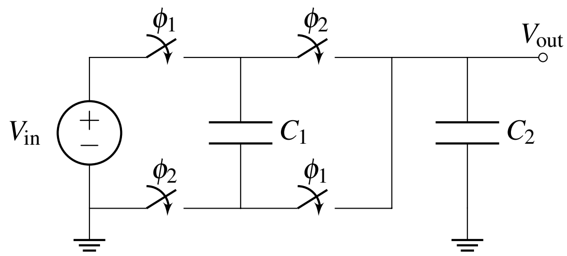
Therefore:

$$q = \frac{1\,\text{V}}{\frac{1}{C_1} + \frac{1}{C_2}}$$

Notice that we have derived the formula for equivalent capacitance of capacitors in series.

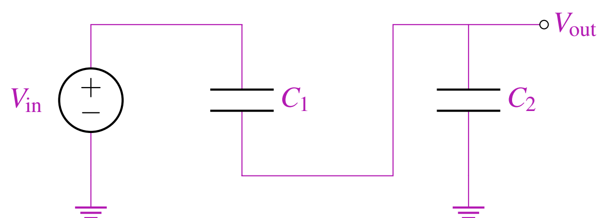
2. Charge Sharing

Consider the circuit shown below. In phase ϕ_1, the switches labeled ϕ_1 are on while the switches labeled ϕ_2 are off. In phase ϕ_2, the switches labeled ϕ_2 are on while the switches labeled ϕ_1 are off.



(a) Redraw the circuit in phase ϕ_1. Label the voltages across each capacitor and find the charge on and voltage across each capacitor as a function of V_{in}, C_1, and C_2. Assume the capacitors are uncharged before phase ϕ_1.

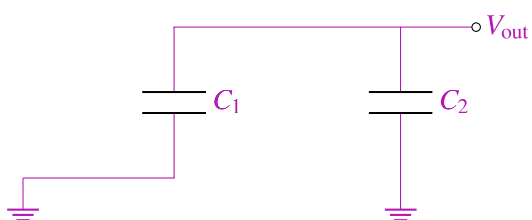
Answer:



The two capacitors in series have a total capacitance of $\frac{C_1C_2}{C_1+C_2}$. We know that there is a voltage of V_{in} across this capacitor and thus $V_{in}\frac{C_1C_2}{C_1+C_2}$ charge. The charge on C_1 must be equal to the charge on C_2. Knowing the charge on each capacitor, we know the voltage across both. Therefore, the voltage across C_1 is $\frac{C_2}{C_1+C_2}V_{in}$. The voltage across C_2 is similarly $\frac{C_1}{C_1+C_2}V_{in}$.

(b) Redraw the circuit in phase ϕ_2. Label the voltages across each capacitor and find the charge on and voltage across each capacitor as a function of V_{out}, C_1, and C_2.

Answer:



The charge on C_1 is simply C_1V_{out}. Similarly the charge on C_2 is C_2V_{out}.

(c) Find V_{out} as a function of V_{in}, C_1, and C_2.

Answer:

We know the total charge in the system is conserved between phase ϕ_1 and phase ϕ_2. There was a charge of $V_{in}\frac{C_1C_2}{C_1+C_2}$ on each capacitor, so the total charge in phase ϕ_1 was $2V_{in}\frac{C_1C_2}{C_1+C_2}$. Since we know charge is conserved, this must be equal to the total charge in phase ϕ_2.

$$2V_{in}\frac{C_1C_2}{C_1+C_2} = (C_1+C_2)V_{out}$$

$$V_{\text{out}} = 2\frac{C_1C_2}{(C_1+C_2)^2}V_{\text{in}}$$

(d) How will the charges be distributed in phase ϕ_2 if we assume $C_1 \gg C_2$?

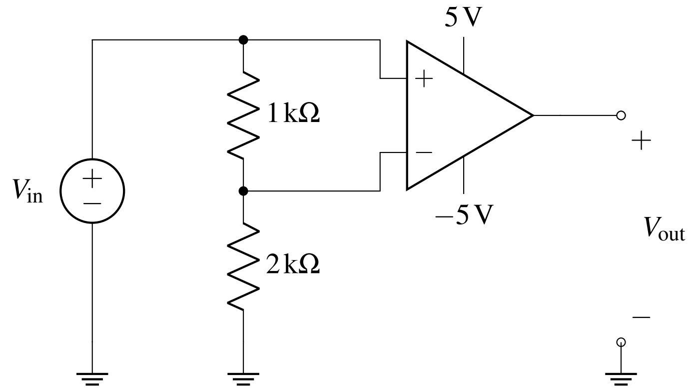
Answer:

We know that the capacitors are in parallel in phase ϕ_2, so the voltage across both capacitors is the same. Considering that $Q = CV$, if $C_1 \gg C_2$, then $Q_1 \gg Q_2$.

3. Op-Amps As Comparators

For each of the circuits shown below, plot V_{out} for V_{in} ranging from $-10\,\text{V}$ to $10\,\text{V}$ for part (a) and from $0\,\text{V}$ to $10\,\text{V}$ for part (b). Let $A = 100$ for your plots. Note that in real op amps, A is typically much higher (i.e. 10^4 – 10^7).

(a)



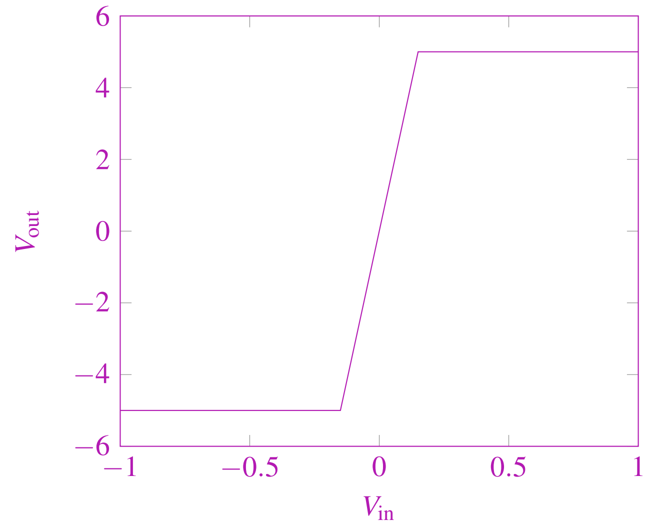
Answer:

$$\begin{aligned}
V_+ &= V_{\text{in}} \\
V_- &= \frac{2\,\text{k}\Omega}{1\,\text{k}\Omega + 2\,\text{k}\Omega}V_{\text{in}} = \frac{2}{3}V_{\text{in}} \\
V_{\text{out}} &= A(V_+ - V_-) + \frac{V_S^+ - V_S^-}{2} + V_S^- \\
&= AV_{\text{in}}\left(1 - \frac{2}{3}\right) + \frac{5-(-5)}{2} + (-5) = \frac{1}{3}AV_{\text{in}}
\end{aligned}$$

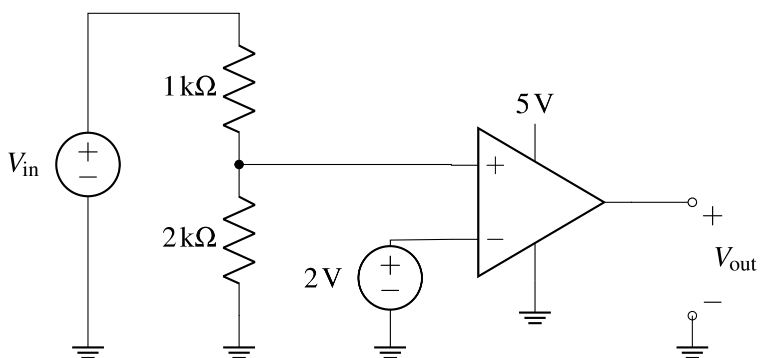
The op-amp satisfies the linear relation above for $V_S^- \leq V_{out} \leq V_S^+$.

$$\begin{aligned}
V_S^- \leq V_{\text{out}} &\leq V_S^+ \\
V_S^- \leq \frac{1}{3}AV_{\text{in}} &\leq V_S^+ \\
3\frac{V_S^-}{A} \leq V_{\text{in}} &\leq 3\frac{V_S^+}{A} \\
3\frac{-5\,\text{V}}{100} \leq V_{\text{in}} &\leq 3\frac{5\,\text{V}}{100} \\
-0.15\,\text{V} \leq V_{\text{in}} &\leq 0.15\,\text{V}
\end{aligned}$$

The op-amp saturates outside of this range.



(b) **[PRACTICE]**



Answer:

$$
\begin{aligned}
V_+ &= \frac{2\,\mathrm{k}\Omega}{1\,\mathrm{k}\Omega + 2\,\mathrm{k}\Omega} V_{\text{in}} = \frac{2}{3} V_{\text{in}} \\
V_- &= 2\,\mathrm{V} \\
V_{\text{out}} &= A(V_+ - V_-) + \frac{V_S^+ - V_S^-}{2} + V_S^- \\
&= A\left(\frac{2}{3}V_{\text{in}} - 2\right) + \frac{5-0}{2} + 0 \\
&= A\left(\frac{2}{3}V_{\text{in}} - 2\right) + 2.5
\end{aligned}
$$

The op-amp satisfies the linear relation above for $V_S^- \leq V_{out} \leq V_S^+$.

$$
\begin{aligned}
V_S^- &\leq V_{\text{out}} \leq V_S^+ \\
V_S^- &\leq A\left(\frac{2}{3}V_{\text{in}} - 2\right) + 2.5 \leq V_S^+ \\
\frac{3}{2}\left(\frac{V_S^- - 2.5}{A} + 2\right) &\leq V_{\text{in}} \leq \frac{3}{2}\left(\frac{V_S^+ - 2.5}{A} + 2\right) \\
\frac{3}{2}\left(\frac{-2.5\,\text{V}}{100} + 2\right) &\leq V_{\text{in}} \leq \frac{3}{2}\left(\frac{5\,\text{V} - 2.5\,\text{V}}{100} + 2\right) \\
2.9625\,\text{V} &\leq V_{\text{in}} \leq 3.0375\,\text{V}
\end{aligned}
$$

The op-amp saturates outside of this range.