
EECS 16A Designing Information Devices and Systems I

Fall 2019 Discussion 14A

1. Orthonormal Matrices and Projections

An orthonormal matrix, \mathbf{A} , is a matrix whose columns, \vec{a}_i , are:

- Orthogonal (ie. $\langle \vec{a}_i, \vec{a}_j \rangle = 0$ when $i \neq j$)
- Normalized (ie. vectors with length equal to 1, $\|\vec{a}_i\| = 1$). This implies that $\|\vec{a}_i\|^2 = \langle \vec{a}_i, \vec{a}_i \rangle = 1$.

- (a) Suppose that the matrix $\mathbf{A} \in \mathbb{R}^{N \times M}$ has linearly independent columns. The vector \vec{y} in \mathbb{R}^N is not in the subspace spanned by the columns of \mathbf{A} . What is the projection of \vec{y} onto the subspace spanned by the columns of \mathbf{A} ?
- (b) Show if $\mathbf{A} \in \mathbb{R}^{N \times N}$ is an orthonormal matrix then the columns, \vec{a}_i , form a basis for \mathbb{R}^N .
- (c) When $\mathbf{A} \in \mathbb{R}^{N \times M}$ and $N \geq M$ (i.e. tall matrices), show that if the matrix is orthonormal, then $\mathbf{A}^T \mathbf{A} = \mathbf{I}_{M \times M}$.
- (d) Again, suppose $\mathbf{A} \in \mathbb{R}^{N \times M}$ where $N \geq M$ is an orthonormal matrix. Show that the projection of \vec{y} onto the subspace spanned by the columns of \mathbf{A} is now $\mathbf{A} \mathbf{A}^T \vec{y}$.

- (e) Given $\mathbf{A} \in \mathbb{R}^{N \times M} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and the columns of \mathbf{A} are orthonormal, find the least squares solution to $\mathbf{A} \hat{\mathbf{x}} = \vec{y}$ where $\vec{y} = [5 \ 12 \ 7 \ 8]^T$.

2. Polynomial Fitting

Let's try an example. Say we know that the output, y , is a quartic polynomial in x . This means that we know that y and x are related as follows:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

We're also given the following observations:

x	y
0.0	24.0
0.5	6.61
1.0	0.0
1.5	-0.95
2.0	0.07
2.5	0.73
3.0	-0.12
3.5	-0.83
4.0	-0.04
4.5	6.42

- (a) What are the unknowns in this question? What are we trying to solve for?
- (b) Can you write an equation corresponding to the first observation (x_0, y_0) , in terms of a_0, a_1, a_2, a_3 , and a_4 ? What does this equation look like? Is it linear in the unknowns?
- (c) Now, write a system of equations in terms of a_0, a_1, a_2, a_3 , and a_4 using *all of the observations*.
- (d) Finally, solve for a_0, a_1, a_2, a_3 , and a_4 using IPython. You have now found the quartic polynomial that best fits the data!