# EECS 16A Designing Information Devices and Systems I
# Fall 2019 Discussion 14B

**1. Orthogonal Matching Pursuit**

Let's work through an example of the OMP algorithm. Suppose that we have a vector $\vec{x} \in \mathbb{R}^4$ that is sparse and we know that it has only 2 non-zero entries. In particular,

$$\mathbf{M}\vec{x} \approx \vec{y} \tag{1}$$

$$\begin{bmatrix} | & | & | & | \\ \vec{m}_1 & \vec{m}_2 & \vec{m}_3 & \vec{m}_4 \\ | & | & | & | \end{bmatrix} \vec{x} \approx \vec{y} \tag{2}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \tag{3}$$

where exactly 2 of $x_1$ to $x_4$ are non-zero. Use Orthogonal Matching Pursuit to estimate $x_1$ to $x_4$.

(a) Why can we not solve for $\vec{x}$ directly?

(b) Why can we not apply the least squares process to obtain $\vec{x}$?

(c) Let us start by reviewing the OMP procedure,

**Inputs:**

- A matrix $\mathbf{M}$, whose columns, $\vec{m}_i$, make up a set of vectors, $\{\vec{m}_i\}$, each of length $n$
- A vector $\vec{y}$ of length $n$
- The sparsity level $k$ of the signal

**Outputs:**

- A vector $\vec{x}$, that contains $k$ non-zero entries.
- A error vector $\vec{e} = \vec{y} - \mathbf{M}\vec{x}$

**Procedure:**

- Initialize the following values: $\vec{e} = \vec{y}$, $j = 1$, $k$, $\mathbf{A} = \begin{bmatrix} \ \end{bmatrix}$
- while ($j \leq k$):
  i. Compute the inner product for each vector in the set, $\vec{m}_i$, with $\vec{e}$: $c_i = \langle \vec{m}_i, \vec{e} \rangle$.
  ii. Column concatenate matrix $\mathbf{A}$ with the column vector that had the maximum inner product value with $\vec{e}$, $c_i$: $\mathbf{A} = \begin{bmatrix} \mathbf{A} & | & \vec{m}_i \end{bmatrix}$
  iii. Use least squares to compute $\vec{x}$ given the $\mathbf{A}$ for this iteration: $\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$
  iv. Update the error vector: $\vec{e} = \vec{y} - \mathbf{A}\vec{x}$
  v. Update the counter: $j = j + 1$

(d) Compute the inner product of every column with the $\vec{y}$ vector. Which column has the largest inner product? This will be the first column of the matrix $\mathbf{A}$.

  

(e) Now, find the projection of $\vec{y}$ onto the columns of $\mathbf{A}$ (ie. $\text{proj}_{\text{Col}(\mathbf{A})}\vec{y} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$). Use this to update the error vector.

(f) Now compute the inner product of every column with the new error vector. Which column has the largest inner product? This will be the second column of $\mathbf{A}$.

(g) We now have two non-zero entries for our vector, $\vec{x}$. Find the values of those two entries.
(Reminder: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)

**2. One Magical Procedure (Fall 2015 Final)**

Suppose that we have a vector $\vec{x} \in \mathbb{R}^5$ and an $N \times 5$ measurement matrix $\mathbf{M}$ defined by column vectors $\vec{c}_1, \ldots, \vec{c}_5$, such that:

$$\mathbf{M}\vec{x} = \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_5 \\ | & & | \end{bmatrix} \vec{x} \approx \vec{b}$$

We can treat the vector $\vec{b} \in \mathbb{R}^N$ as a noisy measurement of the vector $\vec{x}$, with measurement matrix $\mathbf{M}$ and some additional noise in it as well.

You also know that the true $\vec{x}$ is sparse – it only has two non-zero entries and all the rest of the entries are zero in reality. Our goal is to recover this original $\vec{x}$ as best we can.

However, your intern has managed to lose not only the measurements $\vec{b}$ but the entire measurement matrix $\mathbf{M}$ as well!

Fortunately, you have found a backup in which you have all the pairwise inner products $\langle \vec{c}_i, \vec{c}_j \rangle$ between the columns of $\mathbf{M}$ and each other as well as all the inner products $\left\langle \vec{c}_i, \vec{b} \right\rangle$ between the columns of $\mathbf{M}$ and the vector $\vec{b}$. Finally, you also know the inner product of $\vec{b}$ with itself, *i.e.* $\left\langle \vec{b}, \vec{b} \right\rangle$.

All the information you have is captured in the following table of inner products. (These are not the vectors themselves.)

| $\langle \cdot, \cdot \rangle$ | $\vec{c}_1$ | $\vec{c}_2$ | $\vec{c}_3$ | $\vec{c}_4$ | $\vec{c}_5$ | $\vec{b}$ |
|---|---|---|---|---|---|---|
| $\vec{c}_1$ | 2 | 0 | 1 | $-1$ | 1 | 1 |
| $\vec{c}_2$ | | 2 | 1 | $-1$ | $-1$ | $-5$ |
| $\vec{c}_3$ | | | 2 | 0 | $-1$ | 2 |
| $\vec{c}_4$ | | | | 2 | $-1$ | 6 |
| $\vec{c}_5$ | | | | | 2 | $-1$ |
| $\vec{b}$ | | | | | | 29 |

(So, for example, if you read this table, you will see that the inner product $\langle \vec{c}_2, \vec{c}_3 \rangle = 1$, that the inner product $\left\langle \vec{c}_3, \vec{b} \right\rangle = 2$, and that the inner product $\left\langle \vec{b}, \vec{b} \right\rangle = 29$. By symmetry of the real inner product, $\langle \vec{c}_3, \vec{c}_2 \rangle = 1$ as well.)

Your goal is to find which entries of $\vec{x}$ are non-zero and what their values are.

(a) Use the information in the table above to answer which of the $\vec{c}_1, \ldots, \vec{c}_5$ has the largest magnitude inner product with $\vec{b}$.

(b) Let the vector with the largest magnitude inner product with $\vec{b}$ be $\vec{c}_a$. Let $\vec{b}_p$ be the projection of $\vec{b}$ onto $\vec{c}_a$. Write $\vec{b}_p$ symbolically as an expression only involving $\vec{c}_a$, $\vec{b}$, and their inner products with themselves and each other.

(c) Use the information in the table above to find which of the column vectors $\vec{c}_1, \ldots, \vec{c}_5$ has the largest magnitude inner product with the residue $\vec{b} - \vec{b}_p$.

*Hint:* The linearity of inner products might prove useful.

(d) Suppose that the vectors we found in parts (a) and (c) are $\vec{c}_a$ and $\vec{c}_c$. These correspond to the components of $\vec{x}$ that are non-zero, that is, $\vec{b} \approx x_a\vec{c}_a + x_c\vec{c}_c$. However, there might be noise in the measurements $\vec{b}$, so we want to find the least squares estimates $\widehat{x}_a$ and $\widehat{x}_c$. Write a matrix expression for $\begin{bmatrix} \widehat{x}_a \\ \widehat{x}_c \end{bmatrix}$ in terms of appropriate matrices filled with the inner products of $\vec{c}_a$, $\vec{c}_c$, $\vec{b}$.

(e) Compute the numerical values of $\widehat{x}_a$ and $\widehat{x}_c$ using the information in the table.

 