EECS 16A Designing Information Devices and Systems I
Fall 2019 Discussion 2B

There are two different views of matrix-vector multiplication. Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$. The column view of matrix $\mathbf{A}$ can be defined as $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$, where $\vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \in \mathbb{R}^2$ and $\vec{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \in \mathbb{R}^2$. The row view of $\mathbf{A}$ is $\begin{bmatrix} \vec{a}_1' \\ \vec{a}_2' \end{bmatrix}$, where $\vec{a}_1' = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \in \mathbb{R}^2$ and $\vec{a}_2' = \begin{bmatrix} a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^2$. If $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ then the column view of matrix-vector multiplication computes $\mathbf{A}\vec{b}$ as a linear combination of the columns of $\mathbf{A}$,

$$\mathbf{A}\vec{b} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} \vec{b} = b_1\vec{a}_1 + b_2\vec{a}_2 = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{bmatrix}. \tag{1}$$

The row view of matrix-vector multiplication computes $\mathbf{A}\vec{b}$ as a series of inner products,

$$\mathbf{A}\vec{b} = \begin{bmatrix} \vec{a}_1' \\ \vec{a}_2' \end{bmatrix} \vec{b} = \begin{bmatrix} \vec{a}_1'\vec{b} \\ \vec{a}_2'\vec{b} \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{bmatrix}. \tag{2}$$

1. **Span Proofs**

   Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

   (a)
   $$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

   In other words, we can scale our spanning vectors and not change their span.

   (b) **(Practice)**
   $$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$$

   In other words, we can swap the order of our spanning vectors and not change their span.

2. **Visualizing Matrices as Operations**

   This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

   Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices!

   Part 1: Rotation Matrices as Rotations

   (a) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by $15°$ and $30°$, respectively. Design a procedure to rotate the unit square by $45°$ using only $\mathbf{T}_1$ and $\mathbf{T}_2$, and plot the result in the IPython notebook. How would you rotate the square by $60°$?

(b) Try to rotate the unit square by 60° using only one matrix. What does this matrix look like?

(c) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. To do this consider rotating the unit vector $\begin{bmatrix} cos(\alpha) \\ sin(\alpha) \end{bmatrix}$ by θ degrees using the matrix $\mathbf{R}$.

(**Definition:** A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \ldots} = 1$.)

(*Hint: Use your trigonometric identities!*)

(d) **(Practice)** Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition, we will visit inverses very soon in lecture!)

(e) **(Practice)** Use part (d) to obtain the "inverse" rotation matrix for a matrix that rotates a vector by θ. Multiply the inverse rotation matrix with the rotation matrix and vice-versa. What do you get?

(f) What are the matrices that reflect a vector about the (i) x-axis, (ii) y-axis, and (iii) $x = y$

Part 2: Commutativity of Operations

A natural question to ask is the following: Does the *order* in which you apply these operations matter? Follow your TA to obtain the answers to the following questions!

(a) Let's see what happens to the unit square when we rotate the square by 60° and then reflect it along the y-axis.

(b) Now, let's see what happens to the unit square when we first reflect the square along the y-axis and then rotate it by 60°.

(c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

(Practice) Part 3: Distributivity of Operations

(a) The distributivity property of matrix-vector multiplication holds for any vectors and matrices. Show for general $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ that $\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2$.