EECS 16A Fall 2019 Designing Information Devices and Systems I Practice Handout

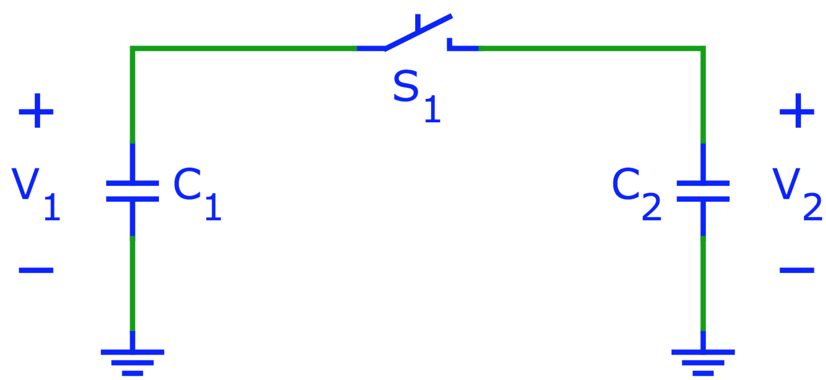
This handout contains a selection of problems that will serve as preparation for Midterm 2.

1. Charge Sharing

In the circuit below, switch S_1 is initially open. Capacitor $C_1 = 10^{-3}$F is initially charged to $V_1 = 3$V and capacitor $C_2 = 3 \times 10^{-3}$F is initially charged to $V_2 = 2$V.

Now S_1 is closed. Calculate the new value of V_2.

Hint: Charge is conserved.



Solution:

Let us define the initial charge on C_1 as Q_{1i} and the initial charge on C_2 as Q_{2i}. We know that $Q_{1i} = C_1 V_{1i}$ and $Q_{2i} = C_2 V_{2i}$, where V_{1i} and V_{2i} are the initial voltages across C_1 and C_2, respectively. (i.e. before switch S_1 is closed). We know from conservation of charge that $Q_{1i} + Q_{2i} = Q_{1f} + Q_{2f}$, where Q_{1f} and Q_{2f} are the final charge on C_1 and C_2. (i.e. after switch S_1 is closed). We can write this as:

(1) $C_1 V_{1i} + C_2 V_{2i} = Q_{1f} + Q_{2f}$.

Additionally, we know that once switch S_1 is closed, the voltage across C_1 and C_2 must be the same, because they are now in parallel with each other. Specifically, $V_{1f} = V_{2f}$ where V_{1f} and V_{2f} are the final voltages across C_1 and C_2, respectively. (i.e. after switch S_1 is closed).

We can write this as:

(2) $V_{1f} = \frac{Q_{1f}}{C_1} = V_{2f} = \frac{Q_{2f}}{C_2}$.

Putting equations (1) and (2) into matrix form, we get the following:

$$\begin{bmatrix} 1 & 1 \\ \frac{1}{C_1} & \frac{-1}{C_2} \end{bmatrix} \begin{bmatrix} Q_{1f} \\ Q_{2f} \end{bmatrix} = \begin{bmatrix} C_1 V_{1i} + C_2 V_{2i} \\ 0 \end{bmatrix}.$$

Plugging in the numbers given to us in the problem we get:

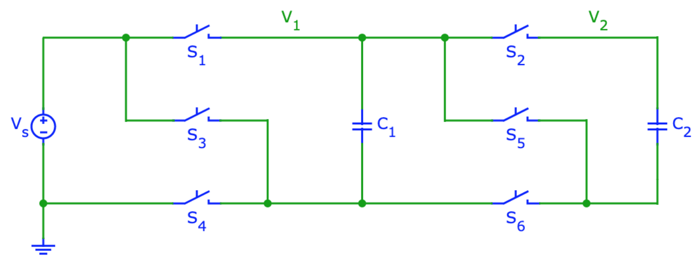
$$\begin{bmatrix} 1 & 1 \\ \frac{1}{0.001} & \frac{-1}{0.003} \end{bmatrix} \begin{bmatrix} Q_{1f} \\ Q_{2f} \end{bmatrix} = \begin{bmatrix} (0.001)(3)+(0.003)(2) \\ 0 \end{bmatrix}.$$

Using either back substitution, Gaussian elimination, or iPython, we find that $Q_{1f} = 0.00225\text{C}$ and $Q_{2f} = 0.00675\text{C}$. We can then calculate $V_{2f} = \frac{Q_{2f}}{C_2} = \frac{0.00675\text{C}}{0.003\text{F}} = 2.25\text{V}$.

(Also, note that $V_{1f} = \frac{Q_{1f}}{C_1} = \frac{0.00225\text{C}}{0.001\text{F}} = 2.25\text{V}$.)

2. Voltage Booster

We have made extensive use of resistive voltage dividers to reduce voltage. What about a circuit that boosts voltage to a value greater than the supply $V_S = 5V$? We can do this with capacitors!



(a) In the circuit above switches S1, S2, S4 and S6 are initially closed and switches S3 and S5 open. Calculate voltages V_1 and V_2.

Solution:

In this setting, the two capacitors in parallel like so:

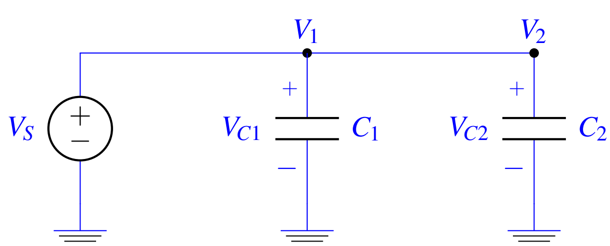


Figure 1: Phase 1

Hence,

$$V_1 = V_2 = V_S = \frac{Q_1}{C_1} = \frac{Q_2}{C_2}.$$

(b) Now, after the capacitors are charged, switches S1, S2, S4 and S6 are opened and switches S3 and S5 closed. Calculate the new voltages V_1 and V_2.

Solution: In phase 2 notice that capacitors C_1 and C_2 have been switched, in a way that the "+" plates are floating i.e. there is no discharge path from nodes V_1 and V_2 to ground. The corresponding schematic is now the following:

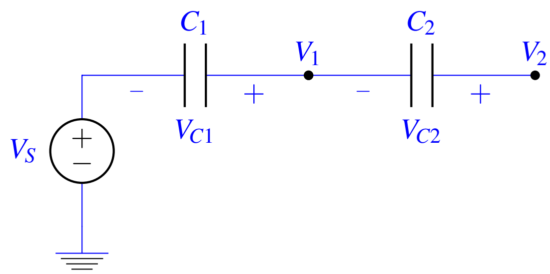


Figure 2: Phase 2

This means that the charge on these plates is going to be preserved so we will have for C_1:

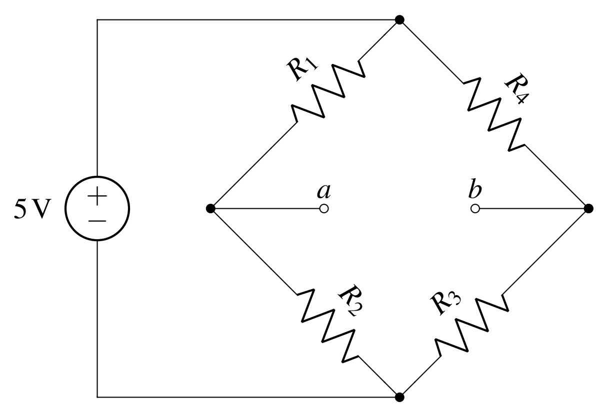
$$Q_1 = V_S C_1 = (V_1 - V_S) C_1 \tag{1}$$
$$V_1 = 2V_S = 10V!! \tag{2}$$

Similarly, for C_2:

$$Q_2 = V_S C_2 = (V_2 - V_1) C_2 \tag{3}$$
$$V_1 = V_S + V_1 = 3V_S = 15V!! \tag{4}$$

3. Wheatstone Bridge

Thévenin equivalence is a powerful technique we can use to analyze the Wheatstone bridge circuit shown below. This circuit is used in many sensor applications where resistors $R_1 - R_4$ are varying with respect to some external actuation. For example, it can be used to build a strain gauge or a scale (remember Fruity Fred from HW 7). In that case the resistors $R_1 - R_4$ would vary with respect to a strain caused by a force, and the Wheatstone Bridge circuit would translate that variation into a voltage difference across the “bridge" terminals a and b. Assume that $R_1 = 2\,\text{k}\Omega$, $R_2 = 2\,\text{k}\Omega$, $R_3 = 1\,\text{k}\Omega$, $R_4 = 4\,\text{k}\Omega$



(a) Calculate the voltage V_{ab} between the two terminals a and b.
Solution:
Notice in the above circuit that there are two voltage dividers, so we can calculate v_a and v_b quickly.

$$v_a = \frac{R_2}{R_1 + R_2} \cdot 5\,\text{V} = 2.5V$$

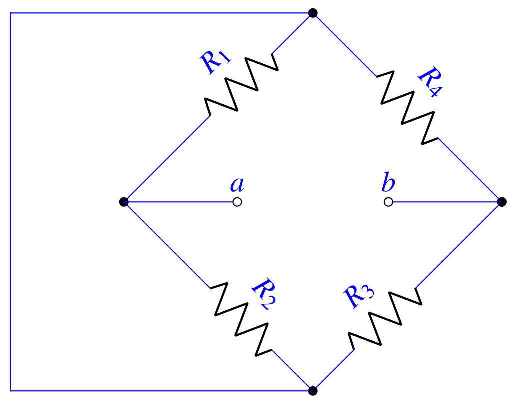
$$v_b = \frac{R_3}{R_3 + R_4} \cdot 5\,\text{V} = 1V$$

Thus, the required voltage difference between the two terminals a and b is: $V_{ab} = v_a - v_b = 1.5V$.

(b) Next, draw the Thévenin equivalent of the Wheatstone bridge circuit.

Solution:

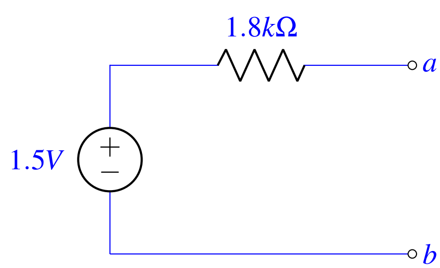
We find the Thévenin resistance by replacing the voltage source with a short and calculating the resistance between the two terminals a and b. The circuit now looks like:



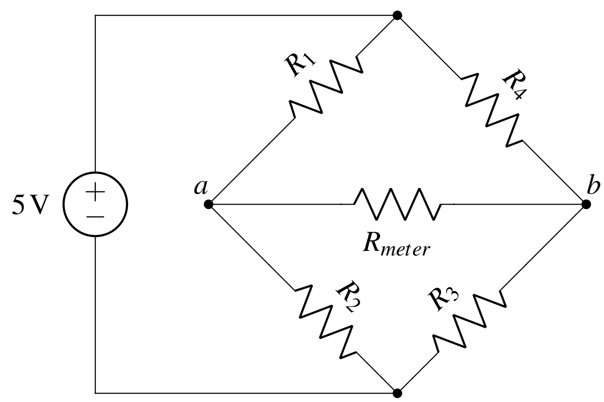
Notice that because the top and bottom node are shorted, we have $R_1 \parallel R_2$ and $R_3 \parallel R_4$. It follows that R_{th} is:

$$\begin{aligned} R_{\text{Th}} &= (R_1 \parallel R_2) + (R_3 \parallel R_4), \text{ where } \parallel \text{ denotes the parallel operator.} \\ &= \frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4} \\ &= 1.8k\Omega \end{aligned}$$

Using $V_{\text{Th}} = V_{ab} = 1.5V$ from part (a), we can construct the Thévenin equivalent circuit:



(c) Now assume that you are trying to measure the voltage V_{ab} using a voltmeter, whose resistance is R_{meter}, so you end up with the circuit below.

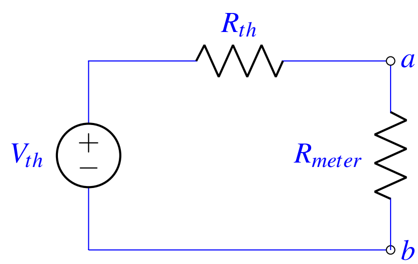


Unfortunately, you voltmeter is far from ideal, so $R_{meter} = 4k\Omega$. Is the voltage V_{ab} you found in part (a) equal to the new voltage $V_{R_{meter}}$ across the voltmeter resistor? Why or why not? Calculate the current $I_{R_{meter}}$ through the voltmeter resistor and the voltage $V_{R_{meter}}$ across the meter resistor.

Solution:

No, the Thévenin voltage we found in part (a) is the open-circuit voltage. If we add R_{meter} back into the original circuit, R_{meter} would load the other resistors (or, equivalently, the Thévenin resistance), so the Thévenin voltage is not equal to the actual voltage across the meter resistor.

Having derived the Thévenin equivalent circuit, we can now draw the following:



Using the facts that, $R_{meter} = 4k\Omega$, $R_{th} = 1.8k\Omega$, $V_{th} = 1.5V$ we can write:

$$I_{R_{meter}} = \frac{1.5V}{1.8k\Omega + 4k\Omega} \approx 0.26\text{mA}$$
$$V_{R_{meter}} = I_{R_{meter}} R_{meter} \approx 1.03V$$

Notice that we have an almost 20% error in our measurement because of the finite impedance of the voltmeter!

4. Digital to Analog Converter (DAC)

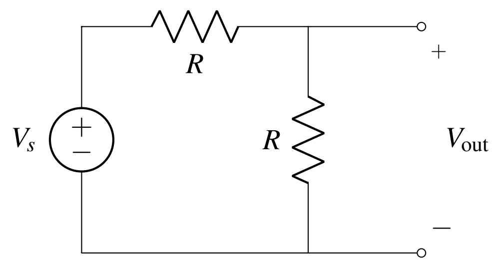
For some outputs, such as audio applications, we need to produce an analog output, or a continuous voltage from 0 to V_s. These analog voltages must be produced from digital voltages, that is sources, that can only be V_s or 0. A circuit that does this is known as a Digital to Analog Converter. It takes a binary representation of a number and turns it into an analog voltage.

The output of a DAC can be represented with the equation shown below:

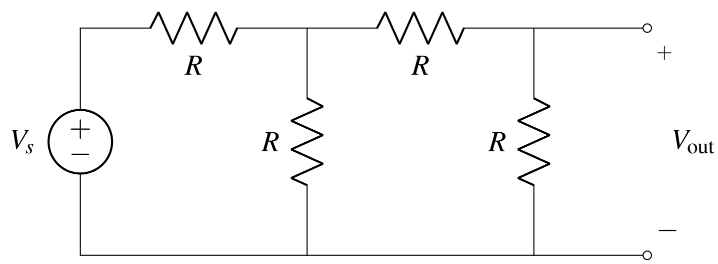
$$V_{\text{out}} = V_s \sum_{n=0}^{N} \frac{1}{2^n} \cdot b_n$$

where each binary digit b_n is multiplied by $\frac{1}{2^n}$.

(a) We know how to take an input voltage and divide it by 2:



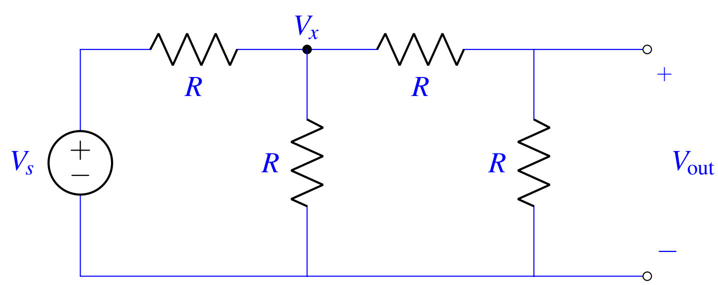
To divide by larger powers of two, we might hope to just "cascade" the above voltage divider. For example, consider:



Calculate V_{out} in the above circut. Is $V_{out} = \frac{1}{4}V_s$?

Solution:

We first find the potential V_x.



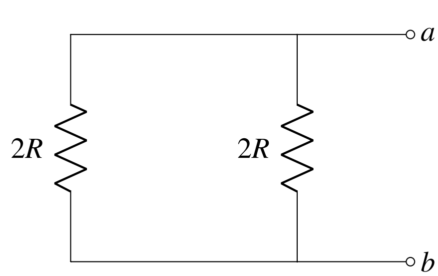
$$V_x = \frac{R \parallel 2R}{R + R \parallel 2R} V_s = \frac{\frac{2}{3}R}{R + \frac{2}{3}R} V_s = \frac{2}{5} V_s$$

$$V_{out} = \frac{R}{R+R} V_x = \frac{1}{2} \cdot \frac{2}{5} V_s = \frac{1}{5} V_s \neq \frac{1}{4} V_s$$

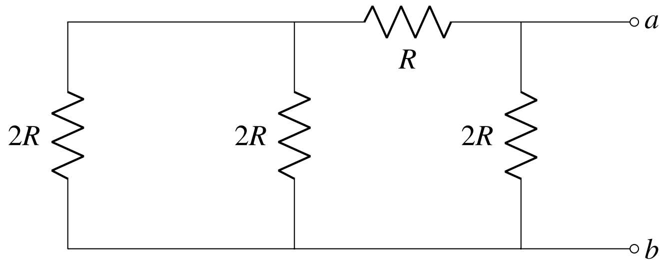
No, V_{out} does not equal $\frac{1}{4}V_s$.

(b) The R-$2R$ ladder, shown below, has a very nice property. For each of the circuits shown below, find the equivalent resistance looking in from points a and b. Do you see a pattern?

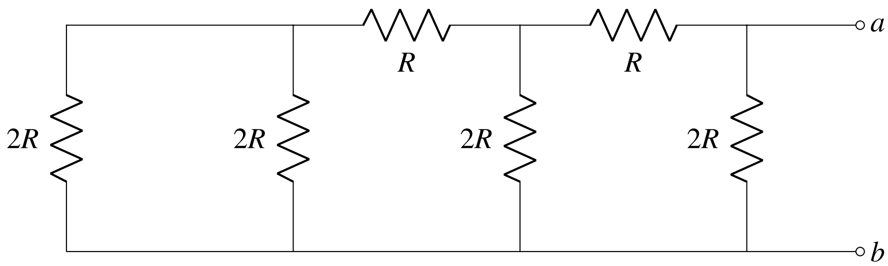
i.



ii.



iii.

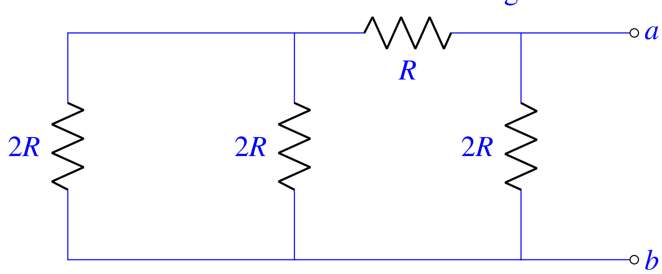


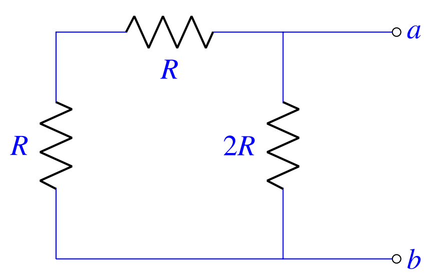
Solution:

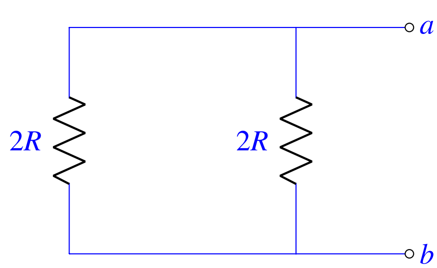
i.

$$R_{eq} = 2R \parallel 2R = R$$

ii. We find the equivalent resistance for the resistors from left to right.

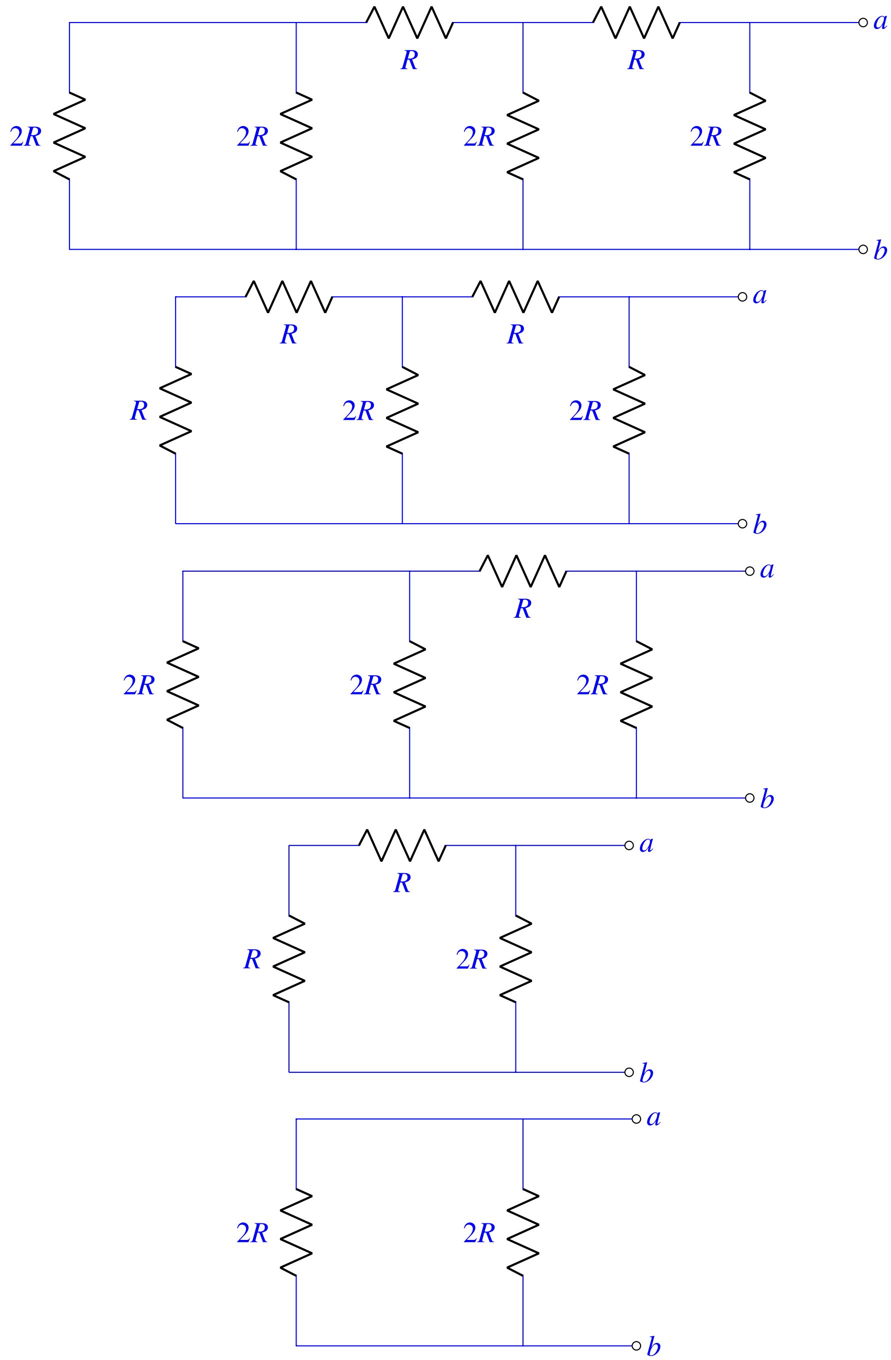






$$R_{eq} = 2R \parallel 2R = R$$

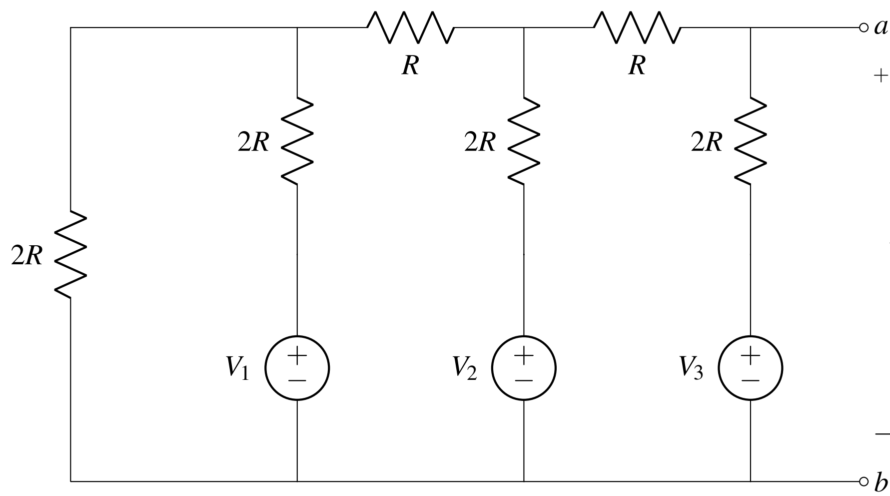
iii. Again, we find the equivalent resistance for the resistors from left to right.



$$R_{eq} = 2R \parallel 2R = R$$

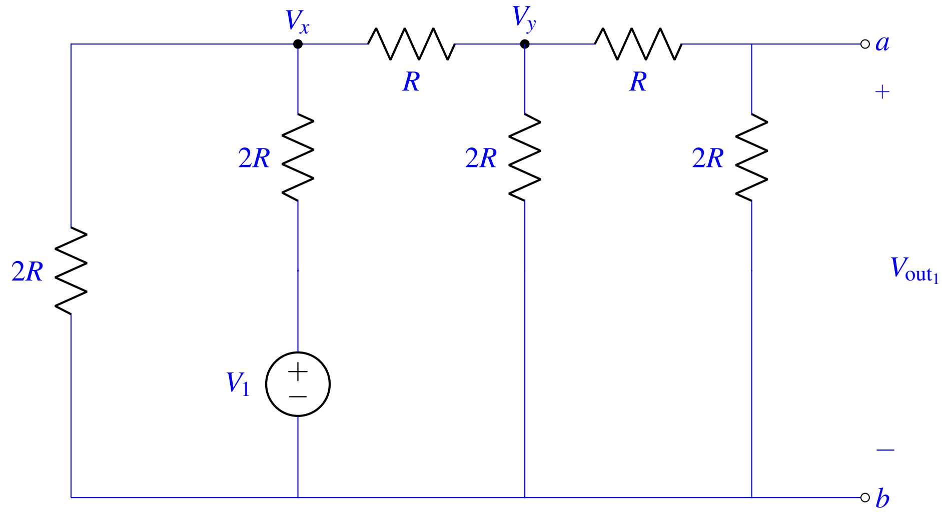
The equivalent resistance is always $R_{eq} = R$.

(c) The following circuit is an R-$2R$ DAC. To understand its functionality, use superposition to find V_{out} in terms of each V_k in the circuit.

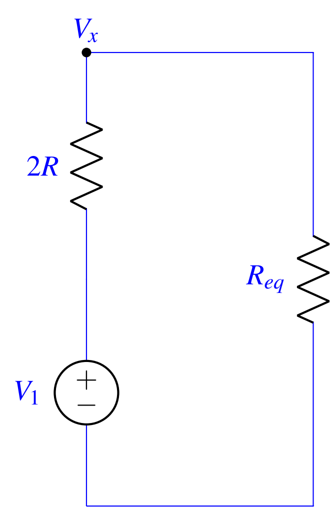


Solution:

V_1:



We first find the potential V_x. To do this, we can simplify the circuit.



$$R_{eq} = 2R \parallel (R + (2R \parallel (R+2R))) = \frac{22}{21}R$$

We can then find V_x using the voltage divider formula.

$$V_x = \frac{\frac{22}{21}R}{2R + \frac{22}{21}R}V_1 = \frac{11}{32}V_1$$

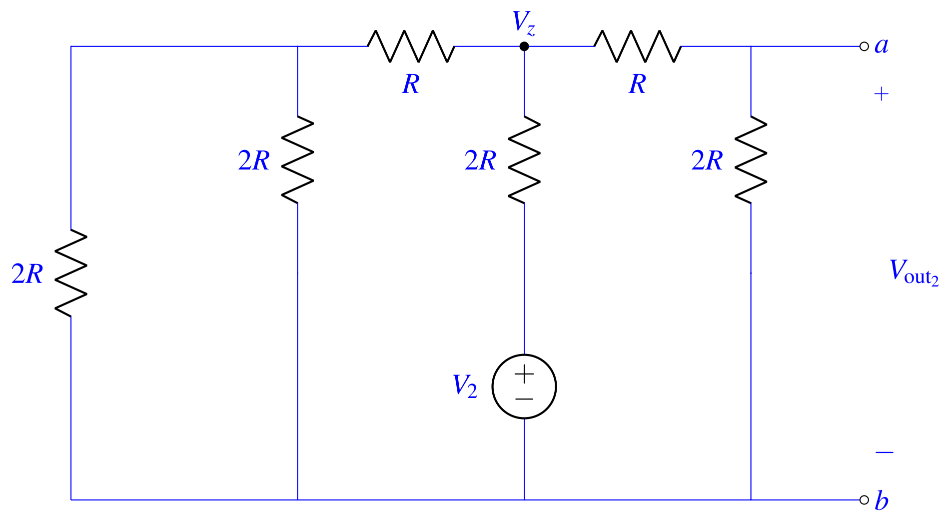
Similarly, we use the voltage divider formula to find V_y in terms of V_x.

$$V_y = \frac{2R \parallel (R+2R)}{R + 2R \parallel (R+2R)}V_x = \frac{\frac{6}{5}R}{R+\frac{6}{5}R}V_x = \frac{6}{11} \cdot \frac{11}{32}V_1 = \frac{3}{16}V_1$$

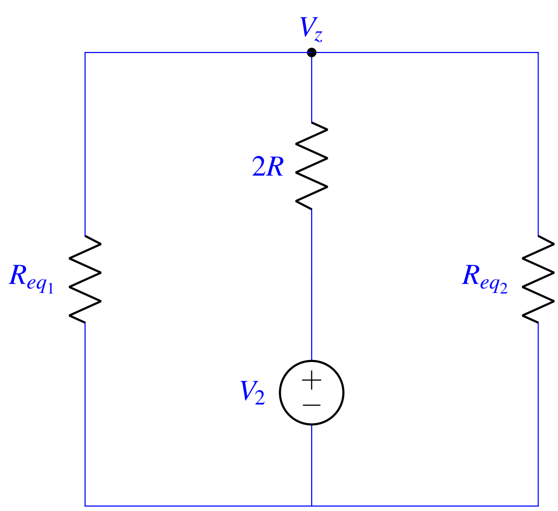
Applying the voltage divider formula again gives us V_{out_1}.

$$V_{\text{out}_1} = \frac{2R}{R+2R}V_y = \frac{2}{3} \cdot \frac{3}{16}V_1 = \frac{1}{8}V_1$$

V_2:



We first find the potential V_z. To do this, we can simplify the circuit.



$$R_{eq_1} = R + (2R \parallel 2R) = R + R = 2R$$
$$R_{eq_2} = R + 2R = 3R$$

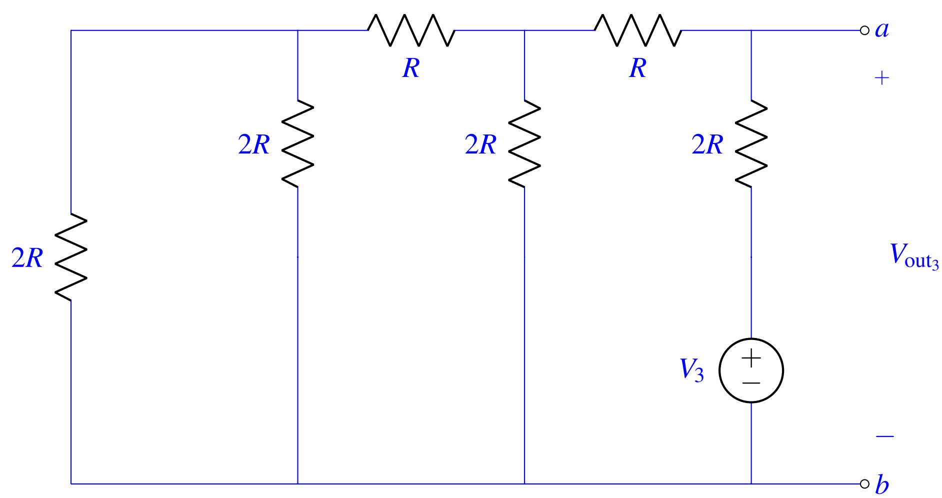
We can then find V_z using the voltage divider formula.

$$V_z = \frac{2R \parallel 3R}{2R + (2R \parallel 3R)} V_2 = \frac{\frac{6}{5}R}{2R + \frac{6}{5}R} V_2 = \frac{3}{8} V_2$$

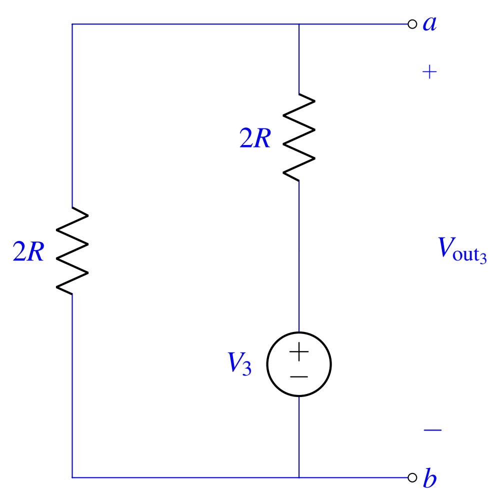
Applying the voltage divider formula again gives us V_{out_2}.

$$V_{\text{out}_2} = \frac{2R}{R + 2R} V_z = \frac{2}{3} \cdot \frac{3}{8} V_2 = \frac{1}{4} V_2$$

V_3:



We can simplify this circuit.



$$V_{\text{out}_3} = \frac{2R}{2R + 2R} V_3 = \frac{1}{2} V_3$$

$$V_{\text{out}} = V_{\text{out}_1} + V_{\text{out}_2} + V_{\text{out}_3} = \frac{1}{8} V_1 + \frac{1}{4} V_2 + \frac{1}{2} V_3$$

(d) We've now designed a 3-bit R-$2R$ DAC. What is the output voltage V_{out} if $V_2 = 1\,\text{V}$ and $V_1 = V_3 = 0\,\text{V}$?

Solution:

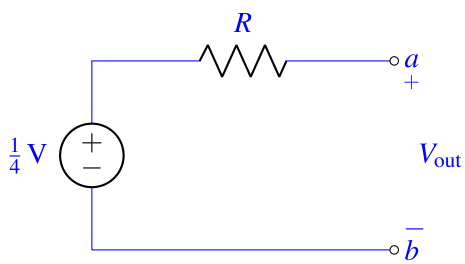
$$V_{\text{out}} = \frac{1}{8}\cdot 0\,\text{V} + \frac{1}{4}\cdot 1\,\text{V} + \frac{1}{2}\cdot 0\,\text{V} = \frac{1}{4}\,\text{V}$$

(e) Draw the Thévenin equivalent of the above circuit, looking in from the terminals a and b with $V_2 = 1\,\text{V}$ and $V_1 = V_3 = 0\,\text{V}$.

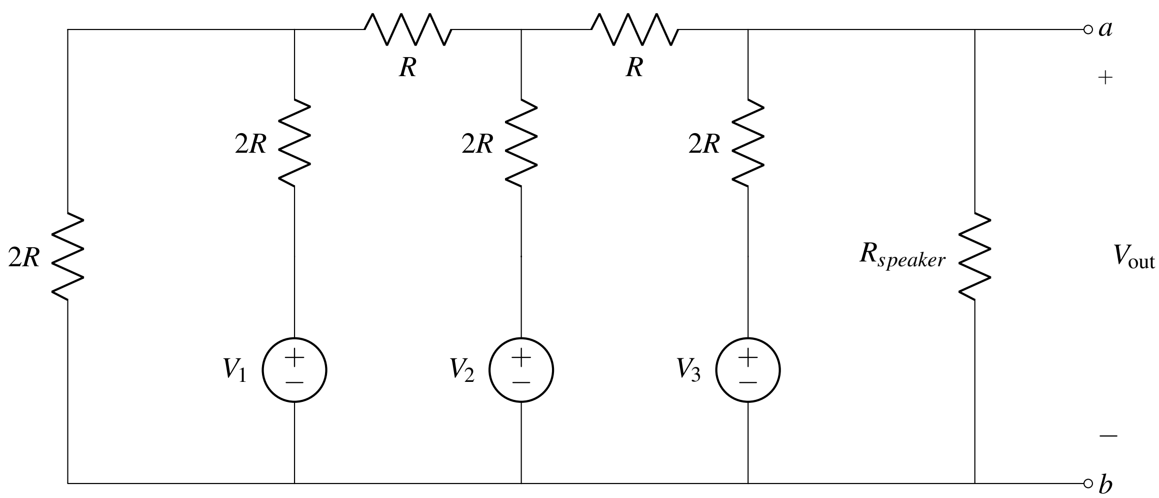
Solution:

$$V_{\text{Th}} = \frac{1}{4}\,\text{V}$$

$$R_{\text{Th}} = R$$

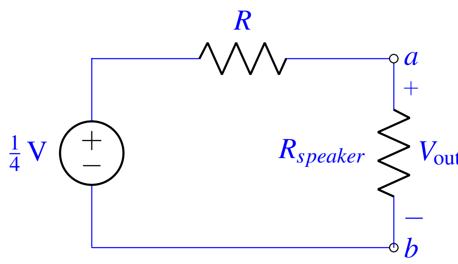


(f) Suppose that we now attach a speaker to the DAC with a resistance of $R_{speaker} = 50\Omega$ as shown in the figure below. Assume also, that the value of R is 50Ω as well. What is the voltage across the speaker and the power dissipated by the speaker? *Hint:* Use the Thevenin equivalent circuit to calculate the above.



Solution:

Attaching the speaker is equivalent to adding a load resistor to the output of the DAC as shown below:



$$V_{\text{out}} = \frac{R_{speaker}}{R + R_{speaker}} \cdot \frac{1}{4}\text{V} = \frac{1}{2} \cdot \frac{1}{4}\text{V} = \frac{1}{8}\text{V}$$

$$P_{\text{out}} = \frac{{V_{\text{out}}}^2}{R_{speaker}} = 0.3125\text{mW}$$

READER Note: The solutions to this problem earlier were $\frac{1}{16}$ V

(g) Repeat part (f) now assuming that the speaker resistance is 100Ω. The value of R remains 50Ω. How do the power and voltage values you found in the two parts compare? Why is the voltage across the speaker in both cases lower than V_{th}?

Solution:

$$V_{\text{out}} = \frac{R_{speaker}}{R + R_{speaker}} \cdot \frac{1}{4}\text{V} = \frac{2}{3} \cdot \frac{1}{4}\text{V} = \frac{1}{6}\text{V}$$

$$P_{\text{out}} = \frac{{V_{\text{out}}}^2}{R_{speaker}} \approx 0.278\text{mW}$$

Notice that even though the voltage across the speaker in this part is (as expected) larger than the voltage in the previous part the power delivered to the speaker is actually smaller! This indicates that maximum power transfer occurs when we have matching i.e. when $R_{speaker} = R_{th}$.
Finally, since the speaker has some equivalent resistance and therefore draws some current from the DAC. As a result, the voltage across the output will be lower than V_{th}.

5. Challenge Problem: Average

The circuit in Figure 3 below operates in time steps k, as illustrated in Figure 4. Each step is of duration T. During each step, switch S_2 is opened, and then switch S_1 is closed immediately afterwards. Then, after time $T/2$, switch S_1 is opened just before switch S_2 is closed.

At the end of each time step the input $V_{in}(kT)$ changes and the process repeats.

Derive an expression for $V_2(kT)$ as a function of $V_{in}(kT)$. Use $C_1 = pC_0$ and $C_2 = (1-p)C_0$.

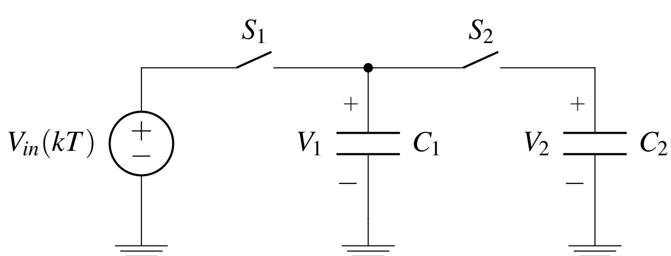


Figure 3: Averaging circuit

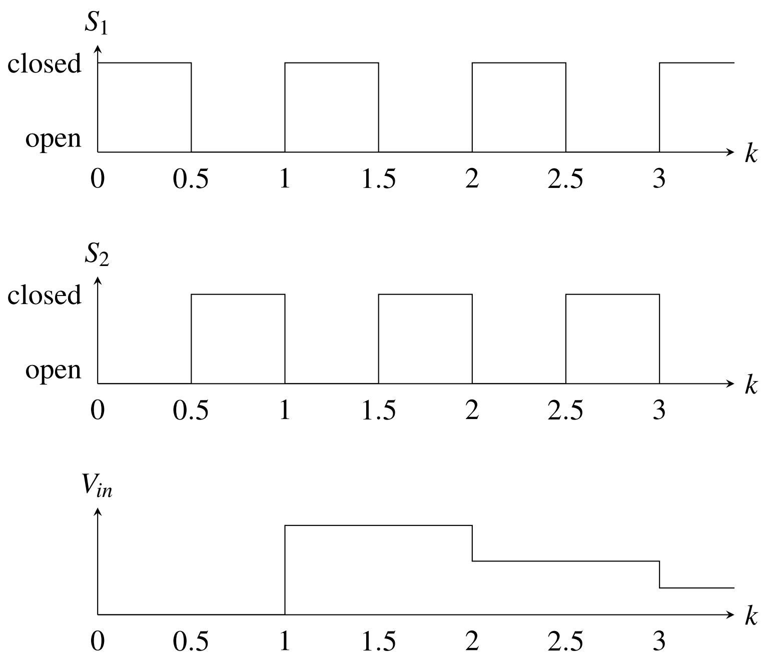


Figure 4: Timing diagram

Solution: Consider the first half of each time step. When switch S_2 is opened, no current can flow across C_2, so its voltage V_2 will remain constant from the previous time step. Then, when switch S_1 is closed, C_1 will be placed in parallel with $V_{in}(kT)$, so it will have a voltage of $V_{in}(kT)$ across its two plates. Thus, the charge on the top plate of C_1 will be $Q = C_1 V_{in}(kT)$ for the first half of the every time step.

Then, when switch S_1 is opened and then S_2 closed, C_1 and C_2 will be placed in parallel with each other, so the charge can flow between the top plate of C_1 and the top plate of C_2.

Notice that the circuit is simply sampling the input during the first half of each cycle and then averaging during the second half of each cycle. Let's see what happens if we apply charge conservation between the first and second half of the first cycle of operation:

$$Q_1(1) + Q_2(1) = Q_1(0.5)$$
$$(C_1 + C_2)V_2(1) = C_1 V_{in}(0.5)$$
$$V_2(1) = \frac{C_1}{C_1 + C_2} V_{in}(0.5)$$
$$V_2(1) = \frac{pC_0 V_{in}(0.5)}{C_0} = pV_{in}(0.5)$$

When we open switch S_2 the charge on C_2 will be preserved, hence the voltage on V_2 will remain unchanged for $k = 1$ to $k = 1.5$ - the circuit has memory! Applying again charge conservation between the first and second half of the second cycle we will get:

$$\begin{aligned}
Q_1(2)+Q_2(2) &= Q_1(1.5)+Q_2(1.5)\\
(C_1+C_2)V_2(2) &= C_1V_{in}(1.5)+C_2V_2(1.5) = C_1V_{in}(1.5)+C_2V_2(1)\\
V_2(2) &= \frac{C_1V_{in}(1.5)+C_2V_2(1)}{C_1+C_2}\\
V_2(2) &= \frac{pC_0V_{in}(1.5)+(1-p)C_0V_2(1)}{C_0}\\
V_2(2) &= pV_{in}(1.5)+(1-p)pV_{in}(0.5)
\end{aligned}$$

A pattern has already started forming - we need to perform a final step of induction to get:

$$V_2(kT) = pV_{in}((k-0.5)T)+(1-p)pV_{in}((k-1.5)T)+(1-p)p^2V_{in}((k-2.5)T)+...+(1-p)^{k-1}V_{in}(0.5T)$$

The circuit calculates an “average” with bigger weights for the most recent values of V_{in} and less weight on the older ones. Pretty cool right?