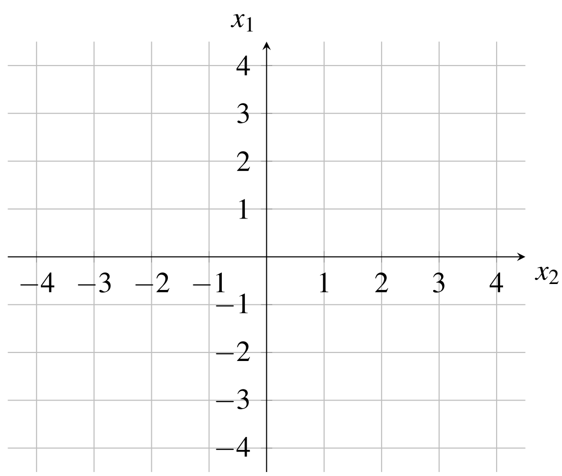
CSM 16A Fall 2020 Designing Information Devices and Systems I Week 12

1. **Projections**

Learning Goal: The goal of this problem is to understand the properties of projection.

Relevant Notes: **Note 23** walks through mathematical derivations for projection.



(a) Consider the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Draw it on the graph provided. Also draw the vector $\vec{y_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Now, find the projections of $\vec{x}$ on $\vec{y_1}$ and $\vec{y_2}$ geometrically. Compare with mathematical calculations.

(b) Calculate the projection of $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ on $\vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Is it the same as the projection of $\vec{y}$ on $\vec{x}$?

(c) Now consider the vectors $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\vec{y_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{y_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Now, find the projections of $\vec{x}$ on $\vec{y_1}$ and $\vec{y_2}$. Also find the projection of $\vec{x}$ on $\text{span}\{\vec{y_1}, \vec{y_2}\}$. Is $\text{proj}_{\vec{y_1}}\vec{x} + \text{proj}_{\vec{y_2}}\vec{x}$ equal to $\text{proj}_{\text{span}\{\vec{y_1},\vec{y_2}\}}\vec{x}$? Explain your answer.

(d) Find the expression for projection of $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ on the columnspace of matrix $\mathbf{A} = \begin{bmatrix} | & | \\ \vec{a_1} & \vec{a_2} \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Is $\text{proj}_{\vec{a_1}}\vec{b} + \text{proj}_{\vec{a_2}}\vec{b}$ equal to $\text{proj}_{\text{Col}\{\mathbf{A}\}}\vec{b}$? (No need to do the calculations.)

If we set up a system of linear equations $\mathbf{A}\vec{x} = \vec{b}$, will there be a unique solution? (No need to solve the system.)

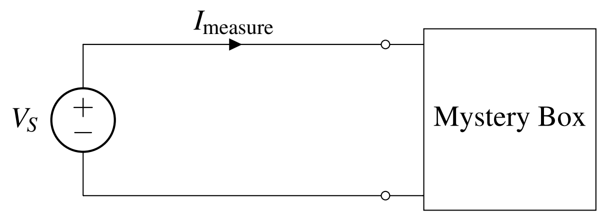
2. And You Thought You Could Ignore Circuits Until Dead Week

Learning Goal: The objective of this problem is to practice solving a noisy system using least squares method.

Relevant Notes: Note 23 covers the details of least squares method.

(a) Write Ohm's Law for a resistor.

(b) You're given the following test setup and told to find R_{eq} between the two terminals of the mystery box. What is R_{eq} of the mystery box between the two terminals in terms of V_S and I_{measure}?



(c) You think you've figured out how to find R_{eq}! You've taken the following measurements:

Measurement #	V_S	I_{measure}
1	2V	1A
2	4V	2A
3	6V	2A
3	8V	4A

Using the information above, formulate a least squares problem whose answer provides an estimate of R_{eq}.

(d) Find the least square error vector $\|\vec{e}\|$.

3. Least Squares Fitting

Learning Goal: The objective of this problem is to set up a least squares problem for coefficients of non-linear equations.

Relevant Notes: Note 23 covers the details of least squares method.

In an upward career move, you join the starship USS Enterprise as a data scientist. One morning the Chief Science Officer, Mr. Spock, hands you some data for the position (y) of a newly discovered particle at different times (t). The data has three points and **contains some noise**:

$$(t=0, y=0.5), \quad (t=1, y=3), \quad (t=2, y=18.5)$$

Your research shows that the path of the particle is represented by the function:

$$y = e^{w_1 + w_2 t} \tag{1}$$

You decide to fit the collected data to the function in Equation (1) using the Least Squares method.

series = qn You need to find the coefficients w_1 and w_2 that *minimize the squared error* between the fitted curve and the collected data points. So you set up a system of linear equations, $\mathbf{A}\hat{\vec{\alpha}} \approx \vec{b}$ in order to find the approximate value of $\hat{\vec{\alpha}} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. What are the values of $\mathbf{A}$ and $\vec{b}$?

seriies = qn Mr. Spock thinks one of the data points is wrong and asks you to redo the fit with only two data points. What will happen to the norm of the error, $\|\vec{e}\| = \left\|\vec{b} - \mathbf{A}\hat{\vec{\alpha}}\right\|$?

seriiies = qn Your colleague tries to repeat your fitting process with the same four data points in part (a), but they misread the equation relating t and y, i.e. they use the following function (which is **different than part (a)**):

$$y = e^{w_1 t + w_2 t} \tag{2}$$

Your colleague tries to find w_1 and w_2 by setting up a system of equations $\mathbf{A}\hat{\vec{\alpha}} \approx \vec{b}$ and utilizing the equation:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \hat{\vec{\alpha}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}. \tag{3}$$

What will happen when your colleague tries to solve the above equation?

4. Besto Pesto (Final Exam, Fall 2018) [PRACTICE]

Your TA Laura is struggling to keep her basil plant alive! She needs your help to determine how much water and sunlight her plant needs.

Let $x_h[k]$ be the plant's height on day k and $x_\ell[k]$ be the number of leaves on the plant on day k. The vector $\vec{x}[k] = \begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix}$ defines the state of the plant. The evolution of the basil plant from one day to the next is defined by the **approximate** mathematical model:

$$\vec{x}[k+1] = \mathbf{A}\vec{x}[k] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix}. \tag{4}$$

(a) Our first goal is to estimate the elements of state transition matrix, $\mathbf{A}$: $a_{11}, a_{12}, a_{21}, a_{22}$. To do this we count the leaves and measure the height for the first N time steps, i.e. we know $\{\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[N]\}$.

Setup a least squares problem to estimate $\vec{a} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$:

$$\hat{\vec{a}} = \underset{\vec{a}}{\operatorname{argmin}} ||\mathbf{M}\vec{a} - \vec{b}||^2. \tag{5}$$

Write the matrix, M, and vector, $\vec{b}$, that would be used in the above least squares problem for $N = 3$.

