CSM 16A Designing Information Devices and Systems I
Fall 2020 Week 1

1. Gaussian Elimination

Learning Goal: The goal of this problem is to use Gaussian Elimination to describe solutions to systems, both qualitatively and quantitatively. Please review **Note 1B** to understand this problem better.

Write each system as an augmented matrix, and then solve using Gaussian Elimination. Also determine whether each system has no solution, a unique solution, or a set of infinitely many solutions.

(a) Solve the following system of equations:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 15 \\ 3x_2 - x_3 &= 8 \\ x_1 + 2x_3 &= 21 \end{aligned}$$

Answer: The system can be written in matrix-vector form:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 21 \end{bmatrix}$$

Then, we write the system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 3 & -1 & 8 \\ 1 & 0 & 2 & 21 \end{array}\right]$$

Here is an example of one possible route taken by Gaussian Elimination (there are many different ways to perform the algorithm):

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 3 & -1 & 8 \\ 1 & 0 & 2 & 21 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 3 & -1 & 8 \\ 0 & 1 & 0 & 6 \end{array}\right] \text{ using } R_3 \leftarrow R_3 - R_1$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 1 & 0 & 6 \\ 0 & 3 & -1 & 8 \end{array}\right] \text{ swapping } R_2 \text{ and } R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -1 & -10 \end{array}\right] \text{ using } R_3 \leftarrow R_3 - 3R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{ using } R_3 \leftarrow -R_3$$

The system is now in **upper triangular matrix** form (row echelon form), and we can see that all three columns have pivot elements of "1" (leading entries 1's). So we can determine that x_1, x_2, and x_3 are all **basic variables**. With three nonzero, consistent equations in this form, the system must have a **unique solution**.

We can either **back-substitute** or continue applying row reductions to generate a **reduced row-echelon form**.

Method I: Back-Substitution: From our augmented matrix in upper triangular form, we have the equations

$$x_1 - x_2 + 2x_3 = 15$$

$$x_2 = 6$$

$$x_3 = 10$$

The latter two already tell us what x_2 and x_3 will be, so we can plug in these values into the first equation to find $x_1 = 15 + 6 - 2(10) = 1$. So the unique solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 10 \end{bmatrix}$$

Method II: Reduced Row-Echelon Form: We continue row reduction to eliminate the non-zero elements *above* the pivots:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 15 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 21 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{ using } R_1 \leftarrow R_1 + R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{ using } R_1 \leftarrow R_1 - 2R_3$$

From our augmented matrix in reduced row echelon form, we have the equations:

$$x_1 = 1$$

$$x_2 = 6$$

$$x_3 = 10$$

So we get the same solution as Method I.

(b) Solve the following system of equations:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_1 + x_3 &= 0 \\ -2x_1 + 2x_2 + x3 &= 5 \\ x_1 + x_2 + 2x_3 &= 2 \end{aligned}$$

Answer: The matrix-vector form can be written as:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 5 \\ 2 \end{bmatrix}$$

We write the system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ -2 & 2 & 1 & 5 \\ 1 & 1 & 2 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -4 \\ 0 & 6 & 7 & 13 \\ 0 & -1 & -1 & -2 \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_1; R_3 \leftarrow R_3 + 2R_1; R_4 \leftarrow R_4 - R_1$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 6 & 7 & 13 \\ 0 & -1 & -1 & -2 \end{array}\right] \text{using } R_2 \leftarrow -R_2/2$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{using } R_3 \leftarrow R_3 - 6R_2; R_4 \leftarrow R_4 + R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{using } R_1 \leftarrow R_1 - 2R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{using } R_1 \leftarrow R_1 - R_3; R_2 \leftarrow R_2 - R_3;$$

The system is now in **upper triangular matrix** form, and we can see that all three columns have pivot elements of "1" (leading entries 1's). So we can determine that x_1, x_2, and x_3 are all **basic variables**, so there is a **unique solution**.

From the first three rows we get:

$$x_1 = -1$$
$$x_2 = 1$$
$$x_3 = 1$$

So we can write the solution set as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(c) (i) Now let us just change the third equation from the last problem to

$$-2x_1 + 2x_2 = 4.$$

The other three equations are unchanged. Do you still have a unique solution?

(ii) What if you change the third equation to

$$-2x_1 + 2x_2 = 5?$$

Answer: (i) We write the system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ -2 & 2 & 0 & 4 \\ 1 & 1 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -4 \\ 0 & 6 & 6 & 12 \\ 0 & -1 & -1 & -2 \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_1; R_3 \leftarrow R_3 + 2R_1; R_4 \leftarrow R_4 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 6 & 6 & 12 \\ 0 & -1 & -1 & -2 \end{array}\right] \text{using } R_2 \leftarrow -R_2/2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{using } R_3 \leftarrow R_3 - 6R_2; R_4 \leftarrow R_4 + R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{using } R_1 \leftarrow R_1 - 2R_2$$

After Gaussian Elimination, we get two rows that look like $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right]$. The system is determined by the other two rows, which represent the equations $x_1 + x_3 = 0$ and $x_2 + x_3 = 2$. Since we have two non-zero rows and three variables, there are **infinitely many solutions**.

The first two columns contain leading 1's or pivot elements. So x_1 and x_2 should be **basic variables**. That leaves x_3 to be the **free variable**.

From the first two rows we get:

$$x_1 + x_3 = 0 \implies x_1 = -x_3$$
$$x_2 + x_3 = 2 \implies x_2 = 2 - x_3$$

So we can write the solution set as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 2 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Alternative approach: We can choose a **parametric representation** of solutions. We choose the free variable as: $x_3 = a$, where a is any scalar. Again the first two rows give us the following:

$$x_1 + a = 0 \implies x_1 = -a$$
$$x_2 + a = 2 \implies x_2 = 2 - a$$

So we can write the solution set as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a \\ 2 - a \\ a \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} a + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Either approach to representing the solution set is fine.

(ii) We write the system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ -2 & 2 & 0 & 5 \\ 1 & 1 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -4 \\ 0 & 6 & 6 & 13 \\ 0 & -1 & -1 & -2 \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_1; R_3 \leftarrow R_3 + 2R_1; R_4 \leftarrow R_4 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 6 & 6 & 13 \\ 0 & -1 & -1 & -2 \end{array}\right] \text{using } R_2 \leftarrow -R_2/2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{using } R_3 \leftarrow R_3 - 6R_2; R_4 \leftarrow R_4 + R_2$$

This system has **no solution**. The third row is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & a \end{array}\right]$, where $a \neq 0$ in the augmented matrix. This row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 1$, which cannot be valid. This means the measurements are not correct, i.e. the system is inconsistent.

(d) (PRACTICE) Solve the following system of equations:

$$\begin{aligned} 2x_2 + 4x_3 &= -2 \\ -5x_3 &= 10 \\ x_1 + x_2 - 3x_3 &= 8 \end{aligned}$$

Answer: The matrix-vector form of the system is:

$$\begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & -5 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 8 \end{bmatrix}$$

Then we write the system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 2 & 4 & -2 \\ 0 & 0 & -5 & 10 \\ 1 & 1 & -3 & 8 \end{array}\right]$$

Notice that the first row already has one 0, the second row has two 0's, and the third row has a nonzero element in the first column. If we swap rows around, this should get us our upper triangular matrix:

$$\left[\begin{array}{ccc|c} 0 & 2 & 4 & -2 \\ 0 & 0 & -5 & 10 \\ 1 & 1 & -3 & 8 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 2 & 4 & -2 \\ 1 & 1 & -3 & 8 \\ 0 & 0 & -5 & 10 \end{array}\right] \text{swapping } R_2 \text{ and } R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 8 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & -5 & 10 \end{array}\right] \text{swapping } R_1 \text{ and } R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 8 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] R_2 \leftarrow R_2/2 \text{ and } R_3 \leftarrow R_3/-5$$

We've reached upper triangular matrix form, and there are three equations with three basic variables, indicating the existence of a **unique solution**.

We can either use back-substitution or further row reduction to find the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

(e) (PRACTICE) Solve the following system of equations:

$$x_1 + 3x_2 - 2x_3 = -3$$
$$2x_1 + 6x_2 - 4x_3 = -5$$

Answer: The equations can be written in matrix-vector form

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

Then we can write the augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -3 \\ 2 & 6 & -4 & -5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right] \text{using } R_2 \leftarrow R_2 - 2R_1$$

This system has **no solution**. The last row is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & a \end{array}\right]$, where $a \neq 0$ in the augmented matrix. This row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 1$, which cannot be valid. This means the measurements are not correct, i.e. the system is inconsistent.

2. Computations: Matrix-Vector Operations

Learning Goal: The goal of this problem is to present various cases of matrix-vector operations such as addition and multiplication. Please review **Note 2A: Section 2.3 and Note 2B: Section 2.1** to understand this problem better.

Consider the following matrices and vectors. Complete the parts below.

$$A = \begin{bmatrix} 2 & 4 \\ 5 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \quad \vec{u_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

(a) What is the transpose of v?

Answer:

$$\vec{v}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

(b) What is $(\vec{v} + \vec{w})^T$? Find $\vec{v}^T + \vec{w}^T$ too. Compare the results.

Answer:

$$\vec{v} + \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 2+(-1) \\ 3+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$$

So $(\vec{v}+\vec{w})^T = \begin{bmatrix}1 & 1 & 7\end{bmatrix}$.
For the second part we have:

$$\vec{v}^T = \begin{bmatrix}1 & 2 & 3\end{bmatrix}$$
$$\vec{w}^T = \begin{bmatrix}0 & -1 & 4\end{bmatrix}$$

Hence $\vec{v}^T + \vec{w}^T = \begin{bmatrix}1 & 1 & 7\end{bmatrix}$. From these results we can see that: $(\vec{v}+\vec{w})^T = \vec{v}^T + \vec{w}^T$

(c) What is $2\vec{v}$ - $4\vec{w}$?

Answer:

$$2\begin{bmatrix}1\\2\\3\end{bmatrix} - 4\begin{bmatrix}0\\-1\\4\end{bmatrix} = \begin{bmatrix}2(1)\\2(2)\\2(3)\end{bmatrix} - \begin{bmatrix}4(0)\\4(-1)\\4(4)\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix} - \begin{bmatrix}0\\-4\\16\end{bmatrix} = \begin{bmatrix}2-0\\4+4\\6-16\end{bmatrix} = \begin{bmatrix}2\\8\\-10\end{bmatrix}$$

(d) What is $\vec{v}^T\vec{w}$?

Answer: The dimension of $\vec{v}^T$ is 1×3 and the dimension of $\vec{w}$ is 3×1. Since the product of an $m\times n$ and an $n\times p$ matrix results in an $m\times p$ matrix, the output in this part is going to be a scalar (i.e. 1×1).

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}0\\-1\\4\end{bmatrix} = \begin{bmatrix}(1)(0)+(2)(-1)+(3)(4)\end{bmatrix} = \begin{bmatrix}0-2+12\end{bmatrix} = \begin{bmatrix}10\end{bmatrix}$$

(e) What is $A\vec{u}_1$? What is $A\vec{u}_2$?

Answer: The dimension of $\mathbf{A}$ is 2×2 and the dimension of $\vec{u}_1$ is 2×1. Therefore, the output in this part is going to have a dimension of 2×1.

$$\mathbf{A}\vec{u}_1 = \begin{bmatrix}2 & 4\\5 & -3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}(2)(1)+(4)(2)\\(5)(1)+(-3)(2)\end{bmatrix} = \begin{bmatrix}2+8\\5-6\end{bmatrix} = \begin{bmatrix}10\\-1\end{bmatrix}$$

$$\mathbf{A}\vec{u}_2 = \begin{bmatrix}2 & 4\\5 & -3\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix} = \begin{bmatrix}(2)(3)+(4)(-4)\\(5)(3)+(-3)(-4)\end{bmatrix} = \begin{bmatrix}6-16\\15+12\end{bmatrix} = \begin{bmatrix}-10\\27\end{bmatrix}$$

(f) What is **AB**? (Do the columns of **AB** look familiar?)

Answer: The dimension of both $\mathbf{A}$ and $\mathbf{B}$ is 2×2. Therefore, the output in this part is going to have a dimension of 2×2.

$$\begin{bmatrix}2 & 4\\5 & -3\end{bmatrix}\begin{bmatrix}1 & 3\\2 & -4\end{bmatrix} = \begin{bmatrix}(2)(1)+(4)(2) & (2)(3)+(4)(-4)\\(5)(1)+(-3)(2) & (5)(3)+(-3)(-4)\end{bmatrix} = \begin{bmatrix}2+8 & 6-16\\5-6 & 15+12\end{bmatrix} = \begin{bmatrix}10 & -10\\-1 & 27\end{bmatrix}$$

We can observe that the columns of AB are the same as the results from the last part. This is because the columns of matrix B are the same as vectors u_1 and u_2, i.e.

$$\mathbf{B} = \begin{bmatrix}| & |\\\vec{u}_1 & \vec{u}_2\\| & |\end{bmatrix}$$
$$\mathbf{AB} = \mathbf{A}\begin{bmatrix}\vec{u}_1 & \vec{u}_2\end{bmatrix} = \begin{bmatrix}\mathbf{A}\vec{u}_1 & \mathbf{A}\vec{u}_2\end{bmatrix}$$

(g) Find $\mathbf{B}^T$. Then express $\mathbf{B}^T$ in terms of $\vec{u}_1$ and $\vec{u}_2$?

Answer:

$$\mathbf{B}^T = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$
$$\mathbf{B}^T = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}^T = \begin{bmatrix} \vec{u}_1{}^T \\ \vec{u}_2{}^T \end{bmatrix}$$

3. Linear or Nonlinear

Learning Goal: The goal of this problem is to draw a distinction between linear and non-linear functions. Please review **Section 1.4 of Note 1** to understand this problem better.

Determine whether the following functions are linear or nonlinear.

(a)
$$f(x_1, x_2) = 3x_1 + 4x_2$$

Answer: To check for linearity, check for **superposition** (additivity) and **homogeneity** (multiplicative scaling). In other words we must check that: $f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2)$, where, α and β are scalars.

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= 3(\alpha x_1 + \beta y_1) + 4(\alpha x_2 + \beta y_2) \\ &= \alpha(3x_1 + 4x_2) + \beta(3y_1 + 4y_2) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

So this function is **linear**. Alternatively you can state that this function is linear because it is of the form:
$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$
where a_1 and a_2 are constants.

(b)
$$f(x_1, x_2) = x_1^2 + e^{x_2}$$

Answer: We check again for superposition and homogeneity.

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= (\alpha x_1 + \beta y_1)^2 + e^{\alpha x_2 + \beta y_2} \\ &= (\alpha x_1)^2 + (\beta y_1)^2 + 2\alpha\beta x_1 y_1 + e^{\alpha x_2} e^{\beta y_2} \\ &\neq \alpha(x_1^2 + e^{x_2}) + \beta(y_1^2 + e^{y_2}) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Hence this function is **not linear**.

(c)
$$f(x_1, x_2) = \sin(a) x_1 + e^b x_2,$$
where a and b are constants.

Answer: Check for superposition and homogeneity:

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= \sin(a)\alpha x_1 + \beta y_1 + \exp(b)(\alpha x_2 + \beta y_2) \\ &= \alpha \sin(a) x_1 + \alpha \exp(b) x_2 + \alpha \sin(a) y_1 + \alpha \exp(b) y_2 \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Hence this function is **not linear**.

(d)

$$f(x_1, x_2) = x_2 - x_1 + 3$$

Answer: Check for superposition and homogeneity:

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= (\alpha x_2 + \beta y_2) - (\alpha x_1 + \beta y_1) + 3 \\ &\neq \alpha(x_2 - x_1 + 3) + \beta(y_2 - y_1 + 3) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

This function is **not linear**, in fact this is an affine function (see **Note 1: Subsection 1.4.2** for more details).

4. Spanning Set

Learning Goal: The goal of this problem is to connect Gaussian Elimination and linear (in)dependence to the concept of span. Another goal is to be comfortable with the geometric representation of span.

(a) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? ("Consistent" means there is at least one solution. Please see **Note 1B: Subsection 1.2.4.2** for more details on consistency of a system.)

$$\mathbf{A}\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Answer: For the system of linear equations to be consistent, there must exist some $\vec{x}$ such that the equality above holds. Performing matrix vector multiplication, we can rewrite the above equality as

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The LHS of the above equation is a linear combination of the columns of $\mathbf{A}$. So, the system will be consistent as long as $\vec{b}$ can be written as a linear combination of the columns of $\mathbf{A}$. In other words, $\vec{b}$ needs to be in range$(\mathbf{A}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$. Please see **Note 3: Section 3.3** for the relation between linear dependence and span.

Note: Any system $\mathbf{A}\vec{x} = \vec{b}$ will be consistent if and only if, $\vec{b} \in$ range$(\mathbf{A})$ i.e. $\vec{b}$ belongs in the span of the columns of A.

Performing Gaussian Elimination on the augmented matrix $[\mathbf{A}|\vec{b}]$, we have:

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 1 & b_2 \\ 0 & 0 & b_3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -3 & b_2-2b_1 \\ 0 & 0 & b_3 \end{array}\right] \text{using } R_2 \leftarrow R_2 - 2R_1$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & (2b_1-b_2)/3 \\ 0 & 0 & b_3 \end{array}\right] \text{using } R_2 \leftarrow -R_2/3$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & (2b_2-b_1)/3 \\ 0 & 1 & (2b_1-b_2)/3 \\ 0 & 0 & b_3 \end{array}\right] \text{using } R_1 \leftarrow R_1 - 2R_2$$

Looking at the last row of the row reduced matrix, that b_3 needs to be zero to avoid inconsistency of the system. On the other hand, solution would exist for any scalar values of b_1 and b_2.

(b) Find out if $\vec{v_1} = \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$ is in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$. What about $\vec{v_2} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$?

Answer: We can use Gaussian Elimination again to find out:

$$\left[\begin{array}{cc|c} 1 & 2 & -3 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & -3 & 11 \\ 0 & 0 & 0 \end{array}\right] \text{using } R_2 \leftarrow R_2 - 2R_1$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 1 & -11/3 \\ 0 & 0 & 0 \end{array}\right] \text{using } R_2 \leftarrow -R_2/3$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 13/3 \\ 0 & 1 & -11/3 \\ 0 & 0 & 0 \end{array}\right] \text{using } R_1 \leftarrow R_1 - 2R_2$$

So we can write that $x_1 = \frac{13}{3}$ and $x_2 = -\frac{11}{3}$.

$$\begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} = \frac{13}{3} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{11}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

So $\vec{v_1}$ is in the span.

If we follow the same process for $\vec{v_2}$, we are going to see that the system is inconsistent, i.e. it cannot be solved for x_1 and x_2, which means $\vec{v_2}$ cannot be in the span.

(c) What is the geometry represented by span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer: span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ contains any vector $\vec{v}$ that can be written as

$$\vec{v} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},$$

where α_1 and α_2 are scalars. We realize that from part (a) and part (b) that any vector whose third component is 0 can be written in this form. Hence, the required span is the set of all vectors that can be written in the form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$. Geometrically, the span is the set of all vector in the xy-plane in $\mathbb{R}^3$.

(d) Reflect on your answer from part(b) and find out span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \right\}$.

Answer: From part (b), we found that $\begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$ is in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$, i.e. $\begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$ is linearly dependent on $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. So any vector $\vec{v} \in$ span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \right\}$ can be written as:

$$\begin{aligned} \vec{v} &= \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \\ &= \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \frac{13}{3}\alpha_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{11}{3}\alpha_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \\ &= (\alpha_1 + \frac{13}{3}\alpha_3) \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (\alpha_2 - \frac{11}{3}\alpha_3) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Since $(\alpha_1 + \frac{13}{3}\alpha_3)$ and $(\alpha_2 - \frac{11}{3}\alpha_3)$ are both scalars in the above equation, we can say that $\vec{v} \in$ span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$. Similarly you can show that any vector $\vec{u} \in$ span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ would also belong in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \right\}$. So span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \right\} =$ span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(e) What is a possible choice for $\vec{v}$ that would make span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v} \right\} = \mathbb{R}^3$?

Answer: From part (a), we realize that any vector whose last component is 0 can be written as a linear combination of the two vectors already in the set. Hence, if we include, for example, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ into the set, then we should be able to reach any vector in $\mathbb{R}^3$. Any vector whose last component is non-zero is a valid addition to the set to achieve the desired span.

Note: We need *at least* n linearly independent vectors $\in \mathbb{R}^n$ to span the entirety of $\mathbb{R}^n$. In other words, span $\{\vec{a}_1, \vec{a}_2, , \ldots, \vec{a}_n\} = \mathbb{R}^n$, for a linearly independent set of $\vec{a}_1, \ldots, \vec{a}_n$.

