

1. Proof on Linear (In)Dependence

Learning Goal: The goal of this problem is to practice some proof development skills.

- (a) **Show that if the system of linear equations, $A\vec{x} = \vec{0}$, has a non-zero solution, then the columns of $A \in \mathbb{R}^{m \times n}$ are linearly dependent.**

We are going to use the approach outlined in [Note 4](#). Please also look into [Note 3 Subsection 3.1.1](#) for the definition of linear dependence/ independence.

- (i) **Start with what we already know:**

We know that system of equations, $A\vec{x} = \vec{0}$, has a non-zero solution, \vec{u} . Express this information in a mathematical form.

- (ii) **Then consider what we need to show:**

We have to show that the columns of A are linearly dependent. Using the definition of linear dependence from [Note 3 Subsection 3.1.1](#), write a mathematical equation that conveys linear dependence of columns of A .

- (iii) **How to go from “what we know” to “what we need to show” :**

Now manipulate the expression from (i) using mathematically logical steps to reach the expression from part (ii).

- (b) **Show that if the system of linear equations: $A\vec{x} = \vec{b}$, has at least one solution for $A \in \mathbb{R}^{m \times n}$, then b should be in the span of the columns of A .**

Please also look into [Note 3 Subsection 3.3](#) for the definition of span.

2. Inverse of a Matrix-Matrix Product

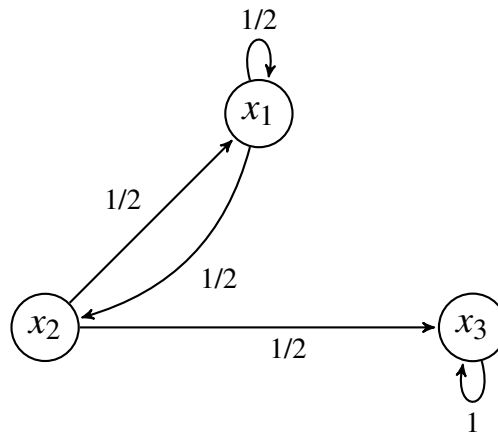
Learning Goal: This problem aims to familiarize you with the properties of inverse and related proof techniques.

Prove that if a matrix-matrix product AB is invertible, the inverse will be equal to $B^{-1}A^{-1}$. Please see [Note 6: subsection 6.1.1](#) for properties of inverse.

3. Functional Pumps

Learning Goal: The goal of this problem is to present a state transition diagram and guide students to understand the meaning of a state transition matrix and its applications. Please review [Note 5: Section 5.1](#) to understand this problem better.

Take a look at this functional pump:



- (a) What do the rows in a functional pump represent? What do the columns represent?
- (b) Analyze the pump above and write the first column of the state transition matrix. Use the state vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

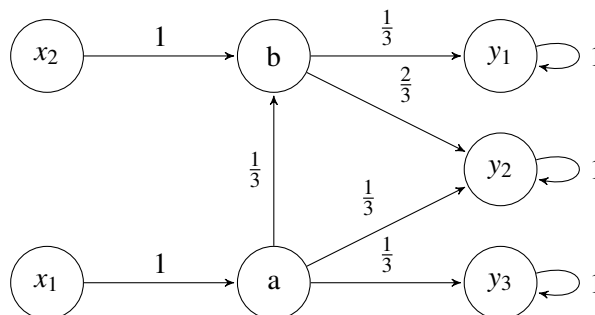
Repeat this process for each of the reservoirs in this diagram.

- (c) Is this system conserved? Why or why not? Please review [Note 5: Section 5.1.4](#) to understand this problem better.

- (d) Given that the initial reservoir volume, $v[0]$, is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ determine the amount of water in each of the reservoirs after turning the system on n number of times. Please review [Note 5: Section 5.1.7](#) to understand this problem better.

- i. Turn the system on once.
- ii. Turn the system on twice.
- iii. What is another way to find $v[2]$ if you could only multiply one state transition matrix into the initial state once?

- (e) **(PRACTICE)** Let us model a system with reservoir states $x_1, x_2, a, b, y_1, y_2, y_3$ as given by the diagram below:



Write the state transition matrix for the above state transition diagram. Use the state vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ a \\ b \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

4. Invertibility and Row Operations

Learning Goal: This question introduces, through the context of finding a given matrix's inverse, how we can represent different types of transformations and row operations with matrices. Also, we will see whether the *order* of applying matrix operations matters. Please review [Section 2.1 of Note 2B](#) and [Section 6.1 of Note 6](#) to understand the problem better.

- (a) Say we have a matrix $\mathbf{M} \in \mathbb{R}^{3 \times n}$ and a matrix \mathbf{A} , which are given by:

$$\mathbf{M} = \begin{bmatrix} \vec{m}_1^T \\ \vec{m}_2^T \\ \vec{m}_3^T \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we left multiply \mathbf{M} by \mathbf{A} (computing the product \mathbf{AM}), what kind of row operation is done on \mathbf{M} ?

- (b) We have the matrix $\mathbf{M} \in \mathbb{R}^{3 \times n}$ as before, as well as the matrix \mathbf{B} , which is given by:

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

If we left-multiply \mathbf{M} by \mathbf{B} , what kind of row operation is done on \mathbf{M} ?

- (c) We have the matrix $\mathbf{M} \in \mathbb{R}^{3 \times n}$ as before, as well as the matrix \mathbf{C} , given by:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What kind of row operation is done on \mathbf{M} ?

- (d) What happens when we apply the transformations (row operations) described in parts (a), (b), and (c)

to the matrix $\mathbf{Q} = \begin{bmatrix} 0 & 0 & 1 \\ -15 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$?

- (e) Multiply the matrices for each of the transformations in parts (a), (b), and (c), so that they are applied in this order: (a) is applied first and (c) is applied last. Call the resulting matrix \mathbf{D} . What happens when you left multiply the \mathbf{Q} from part (d) by \mathbf{D} ? What about right multiplying \mathbf{Q} by \mathbf{D} ? What kind of matrix is \mathbf{D} in relation to \mathbf{Q} ?

- (f) Are there a set of transformations we can apply to $\mathbf{Q} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$ to make it the identity? If so, what are they? If not, why is is not possible?