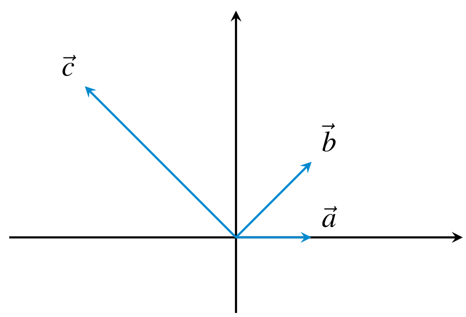
EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 2B

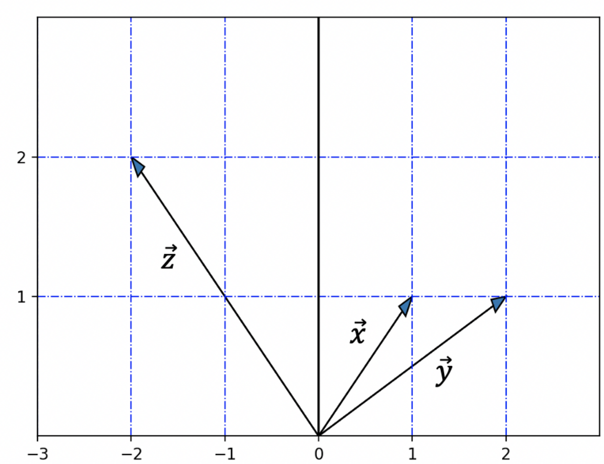
1. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



(a) First, consider the case where $\vec{x}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{y}=\begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{z}=\begin{bmatrix}-2\\2\end{bmatrix}$. Draw these vectors on a sheet of paper.

Answer:



(b) We want to find the two scalars α and β, such that by moving α along $\vec{x}$ and β along $\vec{y}$ so that we can reach $\vec{z}$. Write a system of equations to find α and β in matrix form.

Answer:

$$\alpha\vec{x}+\beta\vec{y}=\vec{z}$$

$$\alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$$

$$\begin{cases} \alpha + \beta \cdot 2 = -2 \\ \alpha + \beta = 2 \end{cases}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(c) Solve for α, β.

Answer: We start by writing the system in the augmented matrix form

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 1 & 1 & 2 \end{array}\right]$$

Then we solve the system using Gaussian Elimination. First, we subtract the second row by the first row:

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -1 & 4 \end{array}\right]$$

Next, we multiply the second row by -1 to solve for β.

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -4 \end{array}\right]$$

We get $\beta = -4$. Then, we take the first row and subtract it by the second row*2.

$$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -4 \end{array}\right]$$

So the solution is $\alpha = 6$ and $\beta = -4$.

2. Finding The Bright Cave

Nara the one-handed druid and Kody the one-handed ranger find themselves in dire straits. Before them is a cliff with four cave entrances arranged in a square: two upper caves and two lower caves. Each entrance emits a certain amount of light, and the two wish to find exactly the amount of light coming from each cave. Here's the catch: after contracting a particularly potent strain of ghoul fever, our intrepid heroes are only able to see the total intensity of light before them (so their eyes operate like a single-pixel camera). Kody and Nara are capable adventurers, but they don't know any linear algebra – and they need your help.

Kody proposes an imaging strategy where he uses his hand to completely block the light from two caves at a time. He is able to take measurements using the following four masks (black means the light is blocked from that cave):

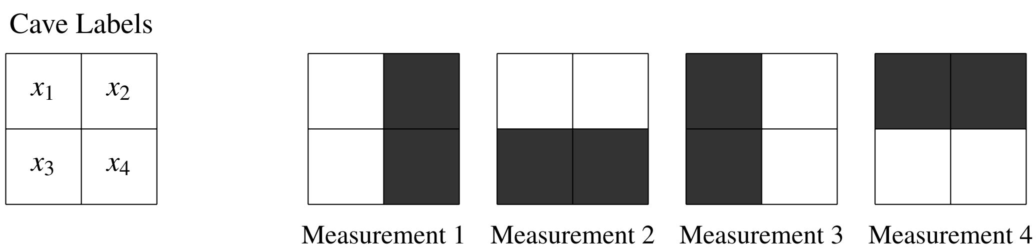


Figure 1: Four image masks.

(a) Let $\vec{x}$ be the four-element vector that represents the magnitude of light emanating from the four cave entrances. Write a matrix $\mathbf{K}$ that performs the masking process in Figure 1 on the vector $\vec{x}$, such that $\mathbf{K}\vec{x}$ is the result of the four measurements.

Answer:

$$\vec{m} = \mathbf{K}\vec{x}$$

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Note here that $\vec{m}$ is the vector of Kody's measurements. The order of the rows does not matter (as long as you tell us which measurement they each correspond to), but the order of the columns does. Re-arranging the columns results in a different set of masks.

(b) Does Kody's set of masks give us a unique solution for all four caves' light intensities? Why or why not?

Answer:

There are two ways to arrive at the answer. We will show both.

i. We can perform Gaussian elimination on the matrix. Now, since we don't know Kody's measurements (the vector $\vec{m}$), we will not augment the matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \tag{1}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_1 - m_2 + m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_1 - m_2 + m_3 \\ 0 & 0 & 0 & 0 & m_4 - m_1 + m_2 - m_3 \end{array}\right] \tag{2}$$

The matrix above has a row of zeroes, which implies that there will either be infinite solutions or no solutions. Therefore, Kody's set of masks cannot give us a unique solution for all four caves' light intensities.

ii. The second way we can show that we will not get a unique solution is to notice the equations. If we find that we could get one equation from the other equations, then we know that the solution is not unique. Notice that the sum of the first and the third row is the same is the sum of the second and fourth row.

$$\begin{aligned} m_1 + m_3 &= m_2 + m_4 \\ m_4 &= m_1 + m_3 - m_2 \\ (x_3 + x_4) &= (x_1 + x_3) + (x_2 + x_4) - (x_1 + x_2) \\ x_3 + x_4 &= x_3 + x_4 \end{aligned}$$

(c) Nara, in her infinite wisdom, places her one hand diagonally across the entrances, covering two of the cave entrances. However, her hand is not wide enough, letting in 50% of the light from the caves covered and 100% of the light from the caves not covered. The following diagram shows the percentage of light let through from each cave:

50%	100%
100%	50%

Does this additional measurement give them enough information to solve the problem? Why or why not?

Answer:

The answer is yes; the additional measurement does give them enough information to solve the problem. Since Nara's measurement is linearly independent from the other four, we are now able to solve for all four light intensities uniquely.

This can be shown using Gaussian elimination with the addition of the following equation:

$$m_5 = \frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4$$

At this point you can either add this equation to make a 5×4 system of equations, or you can remove one of Kody's masks to make a 4×4 system of equations. Here, we write it as a 5×4 matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \\ 0.5 & 1 & 1 & 0.5 & m_5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \\ 0 & 1 & 0.5 & 0.5 & m_5 - \frac{m_1}{2} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 1 & 1 & m_4 \\ 0 & 0 & 1.5 & 0.5 & m_5 + \frac{m_1}{2} - m_2 \end{array}\right] \tag{3}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \\ 0 & 0 & 0 & -1 & m_5 - \frac{3m_3}{2} + \frac{m_2}{2} - m_1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & -1 & m_5 - \frac{3m_3}{2} + \frac{m_2}{2} - m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right] \tag{4}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & 1 & -m_5 + \frac{3m_3}{2} - \frac{m_2}{2} + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 0 & m_5 - \frac{m_3}{2} - \frac{m_2}{2} \\ 0 & 0 & 0 & 1 & -m_5 + \frac{3m_3}{2} - \frac{m_2}{2} + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right] \tag{5}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -m_5 + \frac{m_3}{2} + \frac{m_2}{2} + m_1 \\ 0 & 1 & 0 & 0 & m_5 - \frac{m_3}{2} + \frac{m_2}{2} - m_1 \\ 0 & 0 & 1 & 0 & m_5 - \frac{m_3}{2} - \frac{m_2}{2} \\ 0 & 0 & 0 & 1 & -m_5 + \frac{3m_3}{2} - \frac{m_2}{2} + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right] \tag{6}$$

Notice here that, despite of the row of zeros, we still have four pivot columns. In other words, we have a system of four unkowns and four linearly independent equations. Therefore, we can uniquely determine all four light intensities given Nara's added measurement. However, there may be no solution if $m_4 - m_3 + m_2 - m_1 \neq 0$. Also notice here that the measurements do not determine how we perform our Gaussian elimination.

3. (Optional Practice) Gaussian Elimination

Gaussian Elimination is a systematic procedure that a computer could follow for solving a large system of linear equations simultaneously. The augmented matrix is in the form:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \tag{7}$$

Gaussian Elimination Algorithm for augmented matrix (7):

Step 1: For $i = 1, 2, \cdots, m$

(i) If necessary, swap row i with the row below it, so that the leading entry in row i is as far left as possible.

(ii) Rescale row i so that its leading entry is equal to 1.

(iii) For rows $j = i+1, \cdot, m$, add to row j a scalar multiple of row i, so that the leading entry of row i has all zeros below it.

Step 2: For $i = m, m-1, \cdots, 1$

(i) Add to each row $j = 1, 2, \cdots, i-1$ a scalar multiple of row i so that the leading entry of row i has all zeros above it.

For the following systems, use Gaussian Elimination to solve the problem. Does a solution exist? Is it unique? If there are an infinite number of solutions, give the solution in the parametric form.

(a) Let the three variables be x_1, x_2, x_3. Solve the following augmented matrix form using Gaussian Elimination.

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

Answer:

There are infinite solutions! We use Gaussian elimination to reduce the rows. First we normalize the first row by dividing it with 3.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 2 & 1 \end{array}\right]$$

Then, we divide the second row with 2.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Next, taking $\frac{2}{3}$ times the second row and subtract it from the first row we get

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

The second equation determines that the third unknown, x_3 is $\frac{1}{2}$ and then there are an infinite combination of the first and second unknowns that can satisfy the first equation. To describe this set of solutions, let the second unknown, $x_2 = a$ and $a \in \mathbb{R}$. Then, we solve for the first unknown, x_1, in terms of x_2 using the first equation

$$x_1 - \frac{1}{3}x_2 = 0$$

Therefore we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{a}{3} \\ a \\ \frac{1}{2} \end{bmatrix}$$

(b) True or False: A system of equations with more equations than unknowns will always have either infinite solutions or no solutions.

Answer: False, a counter example of this is when we have N equations and K unknowns ($N > K$) and $N-K$ of the equations are linear combinations of the first K. This means that there are actually K unique equations and K unique unknowns; therefore a unqiue solution will exist. This can be observed when Gaussian elimination is performed and the last $N-K$ rows are all 0.

For example, here is a system of four equations and two unknowns,

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 4 & 2 \\ 2 & 5 & 3 \end{array}\right] \overset{-2R_1+R_3 \to R_3}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 5 & 3 \end{array}\right] \overset{-2R_1-R_2+R_3 \to R_3}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{8}$$

Solving, we get a single exact solution, $x = -1$ and $y = 1$.

(c)

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$

Answer:

In augmented form, the system can be written as

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 6 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 3 \end{array}\right]$$

First we normalize the first row by dividing it with 2.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 3 \end{array}\right]$$

Then, we take subtract the first row from the third row.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

Next, taking two times the second row and subtracting it from the third row.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -6 & 6 \end{array}\right]$$

Then, we divide the third row by -6.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Then, take two times the third row and subtracting it from the second row.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Lastly, we take two times the third row and subtracting it from the first row.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Therefore, the result we get is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 8 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Answer:
Formulate the system into augemented matrix form, we get:

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 1 & 2 & 8 & 0 \\ 1 & 3 & 5 & 3 \end{array}\right]$$

Then, we take the first row and subtract it from the second row.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & -2 & 6 & -2 \\ 1 & 3 & 5 & 3 \end{array}\right]$$

Next, we take the first row and subtract it from the third row.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & -2 & 6 & -2 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

Then, we rescale the second row so that the leading entry is 0. We do this by dividing the second row with -2.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

Then, we add the second row to the third row.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

On the third row, we see that $0x_3 = 2$, which we know there is no solution. Hence, the system has no solutions.

(e)

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

Answer: First we write the system in the augmented matrix form.

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right]$$

Then, we normalize the first row by dividing it with 2.

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right]$$

Next, we take two times the first row and subtract it from the third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -6 \end{array}\right]$$

Then, we take two times the second row and add to the third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The third row of the matrix reduces to 0=0. Therefore, there are infinitely many solutions. It means that for every value of x_3 we choose we can write a value of x_1 and x_2. So let $x_3 = a$ and $a \in \mathbb{R}$. First, we solve for x_2 given the second row:

$$x_2 + x_3 = 3$$

We get that $x_2 = 3 - a$. Then we substitute x_2 and x_3 back into the first row:

$$x_1 + (3 - a) + \frac{3}{2}a = \frac{7}{2}$$

We get $x_1 = \frac{1-a}{2}$. Hence the parametric form of the solution with parameter a is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1-a}{2} \\ 3 - a \\ a \end{bmatrix}$$

