EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 14B

1. Orthonormal Matrices and Projections

An orthonormal matrix, $\mathbf{A}$, is a matrix whose columns, $\vec{a}_i$, are:

- Orthogonal (ie. $\langle \vec{a}_i, \vec{a}_j \rangle = 0$ when $i \neq j$)
- Normalized (ie. vectors with length equal to 1, $\|\vec{a}_i\| = 1$). This implies that $\|\vec{a}_i\|^2 = \langle \vec{a}_i, \vec{a}_i \rangle = 1$.

(a) Suppose that the matrix $\mathbf{A} \in \mathbb{R}^{N \times M}$ has linearly independent columns. The vector $\vec{y}$ in $\mathbb{R}^N$ is not in the subspace spanned by the columns of $\mathbf{A}$. What is the projection of $\vec{y}$ onto the subspace spanned by the columns of $\mathbf{A}$?

Answer: When finding a projection onto a subspace, we're trying to find the "closest" vector in that subspace. This can be found by first finding $\vec{x}$ that minimizes $\|\vec{y} - \mathbf{A}\vec{x}\|$. From least squares, we know that $\vec{\hat{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$. The projection of $\vec{y}$ onto the columns of $\mathbf{A}$ is then $\vec{\hat{y}} = \mathbf{A}\vec{\hat{x}} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$.

(b) Show if $\mathbf{A} \in \mathbb{R}^{N \times N}$ is an orthonormal matrix then the columns, $\vec{a}_i$, form a basis for $\mathbb{R}^N$.

Answer:

We want to show that the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^N$. To show that the columns form a basis for $\mathbb{R}^N$ we need to show two things:

- The columns must form a set of N linearly independent vectors.
- Any vector $\vec{x} \in \mathbb{R}^N$ can be represented as a linear combination of the vectors in the set.

We already know we have N vectors, so first we will show they are linearly independent. We shall do this by showing that $\mathbf{A}\vec{\beta} = \vec{0}$ implies that $\vec{\beta}$ can be only $\vec{0}$.

$$\mathbf{A}\vec{\beta} = \vec{0} \tag{1}$$
$$\beta_1\vec{a}_1 + \ldots + \beta_N\vec{a}_N = \vec{0} \tag{2}$$

Then to exploit the properties of orthogonal vectors, we consider taking the inner product of each side of the above equation with $\vec{a}_i$.

$$\langle \vec{a}_i, \beta_1\vec{a}_1 + \ldots + \beta_N\vec{a}_N \rangle = \langle \vec{a}_i, \vec{0} \rangle = 0 \tag{3}$$

Now we apply the distributive property of the inner product and the definition of orthonormal vectors,

$$\langle \vec{a}_i, \beta_1\vec{a}_1 \rangle + \ldots + \langle \vec{a}_i, \beta_i\vec{a}_i \rangle + \ldots + \langle \vec{a}_i, \beta_N\vec{a}_N \rangle = 0 \tag{4}$$
$$0 + \ldots + \beta_i\langle \vec{a}_i, \vec{a}_i \rangle + \ldots + 0 = 0 \tag{5}$$
$$0 + \ldots + \beta_i\vec{a}_i^T\vec{a}_i + \ldots + 0 = 0 \tag{6}$$

Because $\vec{a}_i^T\vec{a}_i = 1$, $\beta_i = 0$ for the equation to hold. Then, since this is true for all i from 1 to N, all the elements of the vector beta must be zero ($\vec{\beta} = \vec{0}$). Because $\vec{x} = \vec{0}$ implies $\vec{\beta} = \vec{0}$, the columns of $\mathbf{A}$ are linearly independent.

Now, we will show that any vector $\vec{x} \in \mathbb{R}^N$ can be represented as a linear combination of the columns of $\mathbf{A}$.

$$\vec{x} = \mathbf{A}\vec{\beta} = \beta_1\vec{a}_1 + \ldots + \beta_N\vec{a}_N \tag{7}$$

Because we know that the N columns of $\mathbf{A}$ are linearly independent, then there exists $\mathbf{A}^{-1}$. Applying the inverse to the equation above,

$$\mathbf{A}^{-1}\mathbf{A}\vec{\beta} = \mathbf{A}^{-1}\vec{x} \tag{8}$$

$$\vec{\beta} = \mathbf{A}^{-1}\vec{x}, \tag{9}$$

we find that there exists a unqiue β that allow us to represent any $\vec{x}$ as a linear combination of the columns of $\mathbf{A}$.

(c) When $\mathbf{A} \in \mathbb{R}^{N\times M}$ and $N \geq M$ (i.e. tall matrices), show that if the matrix is orthonormal, then $\mathbf{A}^T\mathbf{A} = \mathbf{I}_{M\times M}$.

Answer: Want to show $\mathbf{A}^T\mathbf{A} = \mathbf{I}_{M\times M}$.

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \vec{a}_1^T\vec{a}_1 & \vec{a}_1^T\vec{a}_2 & \ldots & \vec{a}_1^T\vec{a}_M \\ \vec{a}_2^T\vec{a}_1 & \vec{a}_2^T\vec{a}_2 & \ldots & \vec{a}_2^T\vec{a}_M \\ \vdots & \vdots & & \vdots \end{bmatrix} = \mathbf{I}_{M\times M} \tag{10}$$

When $\vec{a}_i^T\vec{a}_i = \|\vec{a}_i\|^2 = 1$ and when $i \neq j$, $\vec{a}_i^T\vec{a}_j = 0$ because the column vectors are orthogonal.

(d) Again, suppose $\mathbf{A} \in \mathbb{R}^{N\times M}$ where $N \geq M$ is an orthonormal matrix. Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of $\mathbf{A}$ is now $\mathbf{A}\mathbf{A}^T\vec{y}$.

Answer:

Starting with the result from part (a),

$$\mathbf{A}\hat{\vec{x}} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}, \tag{11}$$

we can apply the result from part (c),

$$\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y} = \mathbf{A}\mathbf{I}\mathbf{A}^T\vec{y} \tag{12}$$

$$= \mathbf{A}\mathbf{A}^T\vec{y} \tag{13}$$

(e) Given $\mathbf{A} \in \mathbb{R}^{N\times M} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and the columns of $\mathbf{A}$ are orthonormal, find the least squares solution to $\mathbf{A}\hat{\vec{x}} = \vec{y}$ where $\vec{y} = \begin{bmatrix} 5 & 12 & 7 & 8 \end{bmatrix}^T$.

Answer:

Method 1:

Since the columns of $\mathbf{A}$ are orthonormal, from part (d) we know that

$\hat{\vec{x}} = \mathbf{A}^T\vec{y} = \begin{bmatrix} \langle \vec{a}_1, \vec{y} \rangle \\ \langle \vec{a}_2, \vec{y} \rangle \\ \langle \vec{a}_3, \vec{y} \rangle \end{bmatrix}$.

Note that this is equivalent to projecting $\vec{y}$ onto each column of $\mathbf{A}$:

$\hat{x_1} = \frac{\langle \vec{a}_1, \vec{y} \rangle}{||\vec{a}_1||^2} = \langle \vec{a}_1, \vec{y} \rangle = 8$

$\hat{x_2} = \frac{\langle \vec{a}_2, \vec{y} \rangle}{||\vec{a}_2||^2} = \langle \vec{a}_2, \vec{y} \rangle = 7$

$\hat{x_3} = \frac{\langle \vec{a}_3, \vec{y} \rangle}{||\vec{a}_3||^2} = \langle \vec{a}_3, \vec{y} \rangle = \frac{17\sqrt{2}}{2}$

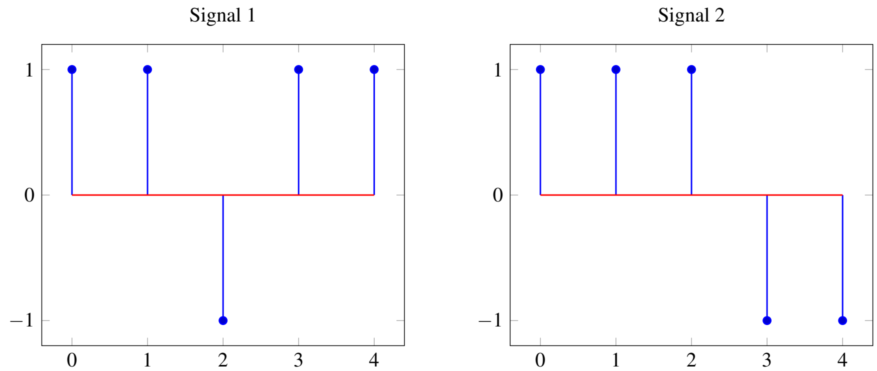
Method 2 (Alternatively you can use the least squares formula):

$$\hat{\vec{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y} = \left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 12 \\ 7 \\ 8 \end{bmatrix}$$

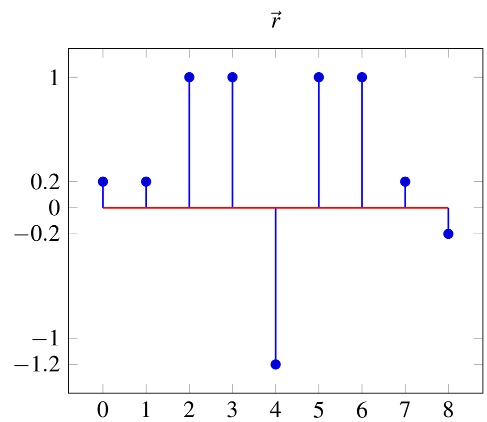
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 12 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ \frac{17\sqrt{2}}{2} \end{bmatrix}$$

2. Identifying satelites and their delays

We are given the following two signals, $\vec{s}_1$ and $\vec{s}_2$ respectively, that are signatures for two satellites.



(a) Your cellphone antenna receives the following signal $r[n]$. You know that there may be some noise present in $r[n]$ in addition to the transmission from the satellite.



Which satellites are transmitting? What is the delay between the satellite and your cellphone? Use cross-correlation to justify your answer. You can use iPython to compute the cross-correlation.

Answer: We calculate both $\text{corr}_r(\vec{s}_1)[k]$ and $\text{corr}_r(\vec{s}_2)[k]$:

$\text{corr}_{\vec{r}}(\vec{s}_1)[k]$

$\vec{r}$	0.2		0.2		1		1		-1.2		1		1		0.2		-0.2	
$\vec{s}_1[n+4]$	1		0		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n+4] \rangle$	0.2	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	$= 0.2$

$\vec{r}$	0.2		0.2		1		1		-1.2		1		1		0.2		-0.2	
$\vec{s}_1[n+3]$	1		1		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n+3] \rangle$	0.2	+	0.2	+	0	+	0	+	0	+	0	+	0	+	0	+	0	$= 0.4$

$\vec{r}$	0.2		0.2		1		1		-1.2		1		1		0.2		-0.2	
$\vec{s}_1[n+2]$	-1		1		1		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n+2] \rangle$	-0.2	+	0.2	+	1	+	0	+	0	+	0	+	0	+	0	+	0	$= 1$

$\vec{r}$	0.2	0.2	1	1	−1.2	1	1	0.2	−0.2	
$\vec{s}_1[n+1]$	1	−1	1	1	0	0	0	0	0	
$\langle \vec{r}, \vec{s}_1[n+1] \rangle$	0.2 +	−0.2 +	1 +	1 +	0 +	0 +	0 +	0 +	0	= 2

$\vec{r}$	0.2	0.2	1	1	−1.2	1	1	0.2	−0.2	
$\vec{s}_1[n]$	1	1	−1	1	1	0	0	0	0	
$\langle \vec{r}, \vec{s}_1[n] \rangle$	0.2 +	0.2 +	−1 +	1 +	−1.2 +	0 +	0 +	0 +	0	= −0.8

$\vec{r}$	0.2	0.2	1	1	−1.2	1	1	0.2	−0.2	
$\vec{s}_1[n-1]$	0	1	1	−1	1	1	0	0	0	
$\langle \vec{r}, \vec{s}_1[n-1] \rangle$	0 +	0.2 +	1 +	−1 +	−1.2 +	1 +	0 +	0 +	0	= 0

$\vec{r}$	0.2	0.2	1	1	−1.2	1	1	0.2	−0.2	
$\vec{s}_1[n-2]$	0	0	1	1	−1	1	1	0	0	
$\langle \vec{r}, \vec{s}_1[n-2] \rangle$	0 +	0 +	1 +	1 +	1.2 +	1 +	1 +	0 +	0	= **5.2**

$\vec{r}$	0.2	0.2	1	1	−1.2	1	1	0.2	−0.2	
$\vec{s}_1[n-3]$	0	0	0	1	1	−1	1	1	0	
$\langle \vec{r}, \vec{s}_1[n-3] \rangle$	0 +	0 +	0 +	1 +	−1.2 +	−1 +	1 +	0.2 +	0	= 0

$\vec{r}$	0.2	0.2	1	1	−1.2	1	1	0.2	−0.2	
$\vec{s}_1[n-4]$	0	0	0	0	1	1	−1	1	1	
$\langle \vec{r}, \vec{s}_1[n-4] \rangle$	0 +	0 +	0 +	0 +	−1.2 +	1 +	−1 +	0.2 +	−0.2	= −1.2

$\vec{r}$	0.2	0.2	1	1	−1.2	1	1	0.2	−0.2	
$\vec{s}_1[n-5]$	0	0	0	0	0	1	1	−1	1	
$\langle \vec{r}, \vec{s}_1[n-5] \rangle$	0 +	0 +	0 +	0 +	0 +	1 +	1 +	−0.2 +	−0.2	= 1.6

$\vec{r}$	0.2	0.2	1	1	−1.2	1	1	0.2	−0.2	
$\vec{s}_1[n-6]$	0	0	0	0	0	0	1	1	−1	
$\langle \vec{r}, \vec{s}_1[n-6] \rangle$	0 +	0 +	0 +	0 +	0 +	0 +	1 +	0.2 +	0.2	= 1.4

$\vec{r}$	0.2	0.2	1	1	−1.2	1	1	0.2	−0.2	
$\vec{s}_1[n-7]$	0	0	0	0	0	0	0	1	1	
$\langle \vec{r}, \vec{s}_1[n-7] \rangle$	0 +	0 +	0 +	0 +	0 +	0 +	0 +	0.2 +	−0.2	= 0

$\vec{r}$	0.2	0.2	1	1	−1.2	1	1	0.2	−0.2	
$\vec{s}_1[n-8]$	0	0	0	0	0	0	0	0	1	
$\langle \vec{r}, \vec{s}_1[n-8] \rangle$	0 +	0 +	0 +	0 +	0 +	0 +	0 +	0 +	−0.2	= −0.2

$\text{corr}_{\vec{r}}(\vec{s}_2)[k]$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n+4]$	−1		0		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+4] \rangle$	−0.2	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= −0.2

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n+3]$	−1		−1		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+3] \rangle$	−0.2	+	−0.2	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= −0.4

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n+2]$	1		−1		−1		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+2] \rangle$	0.2	+	−0.2	+	−1	+	0	+	0	+	0	+	0	+	0	+	0	= −1

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n+1]$	1		1		−1		−1		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+1] \rangle$	0.2	+	0.2	+	−1	+	−1	+	0	+	0	+	0	+	0	+	0	= −1.6

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n]$	1		1		1		−1		−1		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n] \rangle$	0.2	+	0.2	+	1	+	−1	+	1.2	+	0	+	0	+	0	+	0	= 1.6

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-1]$	0		1		1		1		−1		−1		0		0		0	
$\langle \vec{r}, \vec{s}_2[n-1] \rangle$	0	+	0.2	+	1	+	1	+	1.2	+	−1	+	0	+	0	+	0	= 2.4

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-2]$	0		0		1		1		1		−1		−1		0		0	
$\langle \vec{r}, \vec{s}_2[n-2] \rangle$	0	+	0	+	1	+	1	+	−1.2	+	−1	+	−1	+	0	+	0	= −1.2

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-3]$	0		0		0		1		1		1		−1		−1		0	
$\langle \vec{r}, \vec{s}_2[n-3] \rangle$	0	+	0	+	0	+	1	+	−1.2	+	1	+	−1	+	−0.2	+	0	= −0.4

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-4]$	0		0		0		0		1		1		1		−1		−1	
$\langle \vec{r}, \vec{s}_2[n-4] \rangle$	0	+	0	+	0	+	0	+	−1.2	+	1	+	1	+	−0.2	+	0.2	= 0.8

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-5]$	0		0		0		0		0		1		1		1		−1	
$\langle \vec{r}, \vec{s}_2[n-5] \rangle$	0	+	0	+	0	+	0	+	0	+	1	+	1	+	0.2	+	0.2	= 2.4

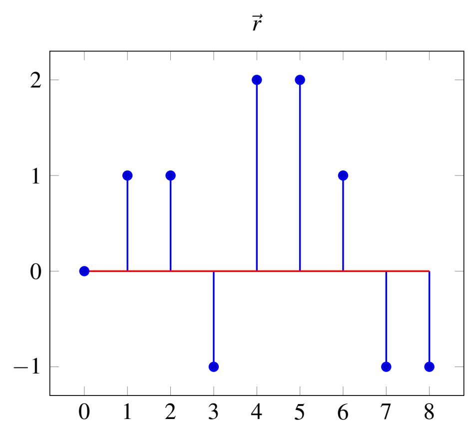
$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-6]$	0		0		0		0		0		0		1		1		1	
$\langle \vec{r}, \vec{s}_2[n-6] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	1	+	0.2	+	−0.2	= 1

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-7]$	0		0		0		0		0		0		0		1		1	
$\langle \vec{r}, \vec{s}_2[n-7] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0.2	+	−0.2	= 0

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-8]$	0		0		0		0		0		0		0		0		1	
$\langle \vec{r}, \vec{s}_2[n-8] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	−0.2	= −0.2

The maximum correlation value is 5.2 at $k = 2$ from satellite 1. Therefore, the transmission likely comes from satellite 1.

(b) Now your cellphone receives a new signal $r[n]$ as below. What the satellites that are transmitting and what is the delay between each satellite and your cellphone?



Answer: We want to find shifts k_1 and k_2 such that: $\vec{r}[n] = \vec{s}_1[n-k_1] + \vec{s}_2[n-k_2]$.
We calculate both $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ and $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ for different shifts k. The index where the maximum correlation value is achieved will tell us the shift indices (delays).

$\text{corr}_{\vec{r}}(\vec{s}_1)[k]$

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_1[n+4]$	1		0		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n+4] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= 0

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_1[n+3]$	1		1		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n+3] \rangle$	0	+	1	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= 1

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n+2]$	-1		1		1		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n+2] \rangle$	0	+	1	+	1	+	0	+	0	+	0	+	0	+	0	+	0	$=2$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n+1]$	1		-1		1		1		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n+1] \rangle$	0	+	-1	+	1	+	-1	+	0	+	0	+	0	+	0	+	0	$=-1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n]$	1		1		-1		1		1		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n] \rangle$	0	+	1	+	-1	+	-1	+	2	+	0	+	0	+	0	+	0	$=1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n-1]$	0		1		1		-1		1		1		0		0		0	
$\langle \vec{r}, \vec{s}_1[n-1] \rangle$	0	+	1	+	1	+	1	+	2	+	2	+	0	+	0	+	0	$=\mathbf{7}$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n-2]$	0		0		1		1		-1		1		1		0		0	
$\langle \vec{r}, \vec{s}_1[n-2] \rangle$	0	+	0	+	1	+	-1	+	-2	+	2	+	1	+	0	+	0	$=1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n-3]$	0		0		0		1		1		-1		1		1		0	
$\langle \vec{r}, \vec{s}_1[n-3] \rangle$	0	+	0	+	0	+	-1	+	2	+	-2	+	1	+	-1	+	0	$=-1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n-4]$	0		0		0		0		1		1		-1		1		1	
$\langle \vec{r}, \vec{s}_1[n-4] \rangle$	0	+	0	+	0	+	0	+	2	+	2	+	-1	+	-1	+	-1	$=1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n-5]$	0		0		0		0		0		1		1		-1		1	
$\langle \vec{r}, \vec{s}_1[n-5] \rangle$	0	+	0	+	0	+	0	+	0	+	2	+	1	+	1	+	-1	$=3$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n-6]$	0		0		0		0		0		0		1		1		-1	
$\langle \vec{r}, \vec{s}_1[n-6] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	1	+	-1	+	1	$=1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n-7]$	0		0		0		0		0		0		0		1		1	
$\langle \vec{r}, \vec{s}_1[n-7] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	-1	+	-1	$=-2$

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_1[n-8]$	0		0		0		0		0		0		0		0		1	
$\langle \vec{r}, \vec{s}_1[n-8] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	−1	= −1

$$\text{corr}_{\vec{r}}(\vec{s}_2)[k]$$

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_2[n+4]$	−1		0		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+4] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= 0

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_2[n+3]$	−1		−1		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+3] \rangle$	0	+	−1	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= −1

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_2[n+2]$	1		−1		−1		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+2] \rangle$	0	+	−1	+	−1	+	0	+	0	+	0	+	0	+	0	+	0	= −2

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_2[n+1]$	1		1		−1		−1		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+1] \rangle$	0	+	1	+	−1	+	1	+	0	+	0	+	0	+	0	+	0	= 1

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_2[n]$	1		1		1		−1		−1		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n] \rangle$	0	+	1	+	1	+	1	+	−2	+	0	+	0	+	0	+	0	= 1

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_2[n-1]$	0		1		1		1		−1		−1		0		0		0	
$\langle \vec{r}, \vec{s}_2[n-1] \rangle$	0	+	1	+	1	+	−1	+	−2	+	−2	+	0	+	0	+	0	= −3

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_2[n-2]$	0		0		1		1		1		−1		−1		0		0	
$\langle \vec{r}, \vec{s}_2[n-2] \rangle$	0	+	0	+	1	+	−1	+	2	+	−2	+	−1	+	0	+	0	= −1

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_2[n-3]$	0		0		0		1		1		1		−1		−1		0	
$\langle \vec{r}, \vec{s}_2[n-3] \rangle$	0	+	0	+	0	+	−1	+	2	+	2	+	−1	+	1	+	0	= 3

$\vec{r}$	0		1		1		−1		2		2		1		−1		−1	
$\vec{s}_2[n-4]$	0		0		0		0		1		1		1		−1		−1	
$\langle \vec{r}, \vec{s}_2[n-4] \rangle$	0	+	0	+	0	+	0	+	2	+	2	+	1	+	1	+	1	= **7**

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n-5]$	0		0		0		0		0		1		1		1		-1	
$\langle \vec{r}, \vec{s}_2[n-5] \rangle$	0	+	0	+	0	+	0	+	0	+	2	+	1	+	-1	+	1	$=3$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n-6]$	0		0		0		0		0		0		1		1		1	
$\langle \vec{r}, \vec{s}_2[n-6] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	1	+	-1	+	-1	$=-1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n-7]$	0		0		0		0		0		0		0		1		1	
$\langle \vec{r}, \vec{s}_2[n-7] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	-1	+	-1	$=-2$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n-8]$	0		0		0		0		0		0		0		0		1	
$\langle \vec{r}, \vec{s}_2[n-8] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	-1	$=-1$

The maximum correlation between signals $\vec{r}$ and $\vec{s}_1$ was achieved at $k_1 = 1$, and the maximum correlation between signals $\vec{r}$ and $\vec{s}_2$ was achieved at $k_2 = 4$.