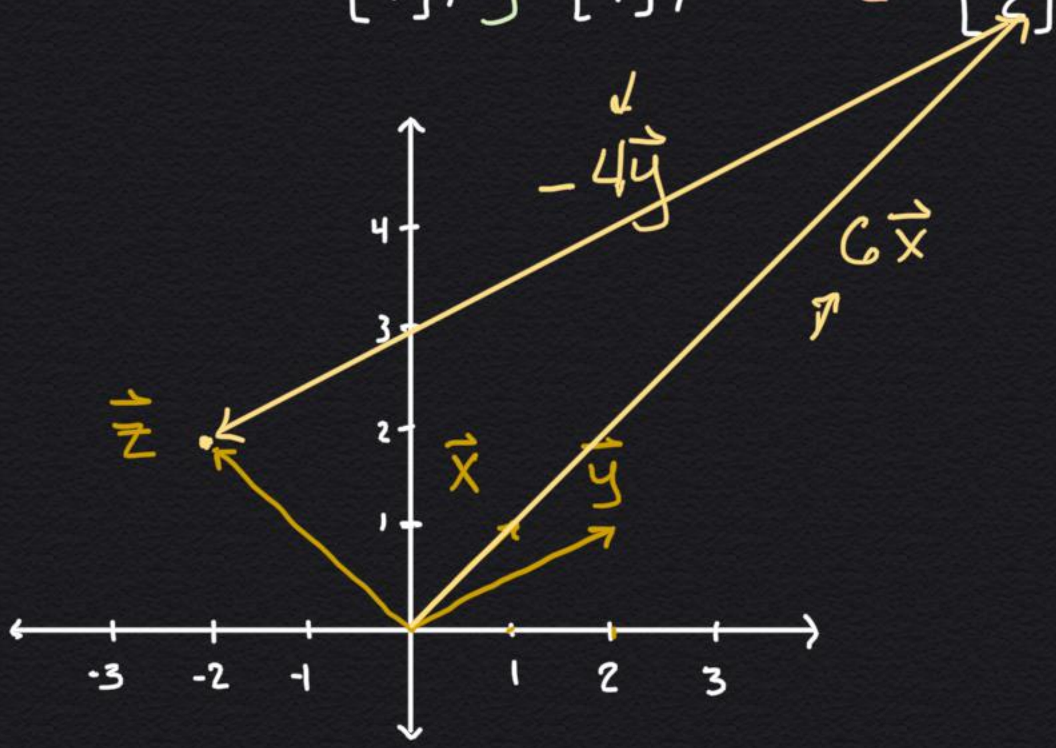
① Visualizing Span: Finding $\alpha, \beta \in \mathbb{R}$ so that $\alpha\vec{a} + \beta\vec{b} = \vec{c}$

a) Draw: $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{y} = \begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{z} = \begin{bmatrix}-2\\2\end{bmatrix}$



$$6\vec{x} - 4\vec{y} = \vec{z}$$

$\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{y} = \begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{z} = \begin{bmatrix}-2\\2\end{bmatrix}$

b/c) Solve for $\alpha, \beta \in \mathbb{R}$ so that $\alpha\vec{x} + \beta\vec{y} = \vec{z}$

(b)

$$\alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix}\alpha\\\alpha\end{bmatrix} + \begin{bmatrix}2\beta\\\beta\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix}\alpha + 2\beta\\\alpha + \beta\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$$

Matrix Form!

(c)

$R_2 \rightarrow R_2 - R_1$

$$\left[\begin{array}{cc|c}1 & 2 & -2\\ \cancel{1-1} & \cancel{1-2} & \cancel{2-(-2)}\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}1 & 2 & -2\\0 & -1 & 4\end{array}\right]$$

(0, −1, 4)

$\Downarrow$ $R_2 \rightarrow -R_2$

$$\left[\begin{array}{cc|c}1 & 2 & -2\\0 & 1 & -4\end{array}\right]$$

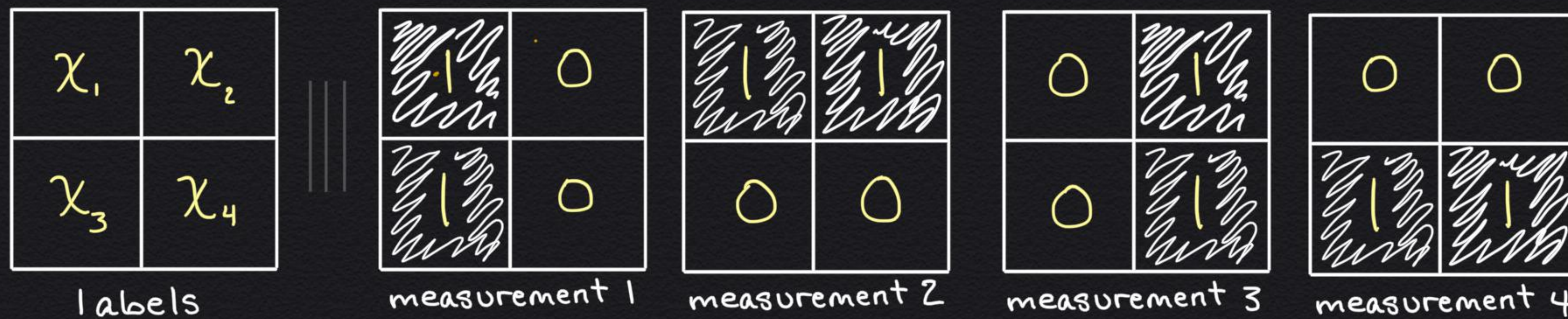
$$\boxed{\beta = -4}$$

$$1\cdot\alpha + 2\cdot\cancel{(-4)}^{-8} = -2$$

$$\alpha = -2 + 8$$

$$\boxed{\alpha = 6}$$

② The Cave (Nara & Cody): Find the light from each cave!



a) $\vec{x}$ labels the cave lighting.
Write out a matrix K, so that $K\vec{x}$ performs the masking process.

$$x_1 + x_3 = m_1$$
$$x_1 + x_2 = m_2$$
$$x_2 + x_4 = m_3$$
$$x_3 + x_4 = m_4$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}$$

$K \quad \vec{x} = \vec{m}$

"Casual" Method: (b)

$$x_1 + x_2 + x_3 + x_4 = m_1 + m_3$$
$$x_1 + x_2 + x_3 + x_4 = m_2 + m_4$$

$$[\,0 = m_1 - m_2 + m_3 - m_4\,]$$

b) Can we get a unique solution?

- No solution, if $m_1 - m_2 + m_3 - m_4 \neq 0$
- Otherwise there are infinite solutions ∞

Formal Method:

(Augmented Form)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \xrightarrow[R_2 \to R_2 - R_1]{} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right]$$

$R_4 \to R_4 - R_3$

Final Form

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right]$$

$$0 = m_1 - m_2 + m_3 - m_4$$

No unique solution!

x_1	x_2
x_3	x_4

0.5	1.0
1.0	0.5

measurement 5

C) Suppose Nara makes a 5^{th} measurement. Now can we determine the light from each cave?

$$\left[\tfrac{1}{2}x_1 + x_2 + x_3 + \tfrac{1}{2}x_4 = m_5\right]$$

$$K^{(5\times4)}\,\vec{x}^{(4\times1)} = \vec{m}^{(5\times1)}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1/2 & 1 & 1 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \\ 1/2 & 1 & 1 & 1/2 & m_5 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \\ 1/2 & 1 & 1 & 1/2 & m_5 \end{array}\right] \xrightarrow{R_5 \to R_5 - \frac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \\ 0 & 1 & 1/2 & 1/2 & m_5 - \frac{1}{2}m_1 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 1 & 1 & m_4 \\ 0 & 1 & 1/2 & 1/2 & m_5 - \frac{1}{2}m_1 \end{array}\right] \xrightarrow{R_5 \to R_5 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 1 & 1 & m_4 \\ 0 & 0 & 3/2 & 1/2 & m_5 - \frac{1}{2}m_1 - m_2 + m_1 \end{array}\right] \xrightarrow{R_4 \to R_4 - R_3}$$

$m_5 + \frac{1}{2}m_1 - m_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \\ 0 & 0 & 3/2 & 1/2 & m_5 + \frac{1}{2}m_1 - m_2 \end{array}\right] \xrightarrow{R_4 \leftrightarrow R_5} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 3/2 & 1/2 & m_5 + \frac{1}{2}m_1 - m_2 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right]$$

$R_4 \to R_4 - \frac{3}{2}R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & -1 & m_5 - m_1 + \frac{1}{2}m_2 - \frac{3}{2}m_3 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right] \xrightarrow{R_4 \to -R_4} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & 1 & -m_5 + m_1 - \frac{1}{2}m_2 + \frac{3}{2}m_3 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right]$$

$$x_4 = -m_5 + m_1 - \tfrac{1}{2}m_2 + \tfrac{3}{2}m_3$$

$$x_3 + (\quad) = m_3 - m_2 + m_1$$

③ Gaussian Elimination Practice

a)
$$\left[\begin{array}{ccc|c} \underline{3} & -1 & 2 & 1 \\ 0 & 0 & \underline{\underline{2}} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1/3 & 2/3 & 1/3 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

b) True/False: There is never a unique solution if the number of equations don't match.

False! Our problem 2c above is a counter example!
Also, think of this...

It's clear that $x=1$ & $y=2$, but there are 3 equations!!

$$\begin{aligned} x &= 1 \\ y &= 2 \\ 2y &= 4 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right]$$

c)
$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & -2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -6 & 6 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \checkmark$$

Unique Solution

d)

$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 8 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & -2 & 6 & -2 \\ 1 & 3 & 5 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 1 & 3 & 5 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$ No Solution!!!

e)

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 7/2 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 7/2 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 7/2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∞ solutions!
(infinite)