Discussion 3A

1] Span Proofs

(a) Prove $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ for $\alpha \in \mathbb{R}$, and $\alpha \neq 0$:

$$\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \equiv S_1$$

$$\vec{r} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = S_2$$

If $\alpha = 0$

$S_2 \subseteq S_1$

But! $S_1 \not\subseteq S_2$

1. $\vec{q} \in S_2 \rightarrow S_1 \subseteq S_2$
2. $\vec{r} \in S_1 \rightarrow S_2 \subseteq S_1$

Means: "span 2 is contained in span 1"

$S_1 = S_2$

Q.E.D.

1] $\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n$

$= \left(\frac{a_1}{\alpha}\right)(\alpha\vec{v}_1) + a_2\vec{v}_2 + \dots + a_n\vec{v}_n \in S_2$

2] $\vec{r} = b_1(\alpha\vec{v}_1) + b_2\vec{v}_2 + \dots + b_n\vec{v}_n$

$= (\alpha b_1)\vec{v}_1 + b_2 v_2 + \dots + b_n\vec{v}_n \in S_1$

Note! Must show both ways!!

For instance, if $\alpha = 0$ we note S_2 is a smaller span and the spans are not equal. We see proof part 1 fails since $(a_1/\alpha) \rightarrow \infty$ so we can find that term.

2] Visualizing Matrices (as operations):

Part 1: Rotations

a] Given T_1, T_2 matrices (which rotate plane by 15° and 30°),
Show how to rotate $\vec{x}$ by 45°, and by 60°:

b] What does the 60° rotation matrix look like?

$T_1\vec{x}$ ~ "rotate 15°"

$T_2\vec{x}$ ~ "rotate 30°"

$T_2 = T_1T_1$

$$T_2(T_1\vec{x}) \sim \text{"rotating } 45°\text{"} = \overbrace{(T_2T_1)}^{T_{45°}}\vec{x} = (T_1T_2)\vec{x}$$

$$(T_2T_2) = T_1T_1T_1T_1 = T_1T_1T_2 = \quad = T_{60°}$$

Note! There are many solutions for these matrice because T_1 and T_2 commute $T_1T_2 = T_2T_1$, but this is not true for general matrices.

c) A general rotation by 'θ' is given by $R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

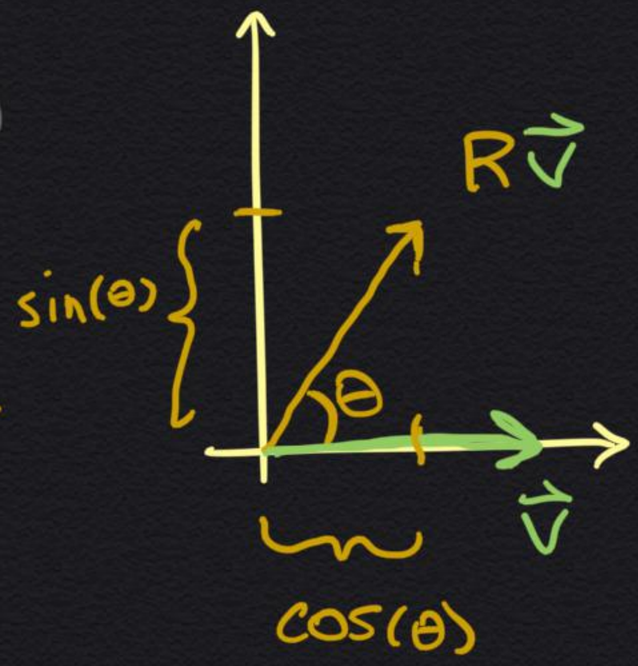
Show this is true by rotating $\vec{x} = \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$.

Hint: $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$

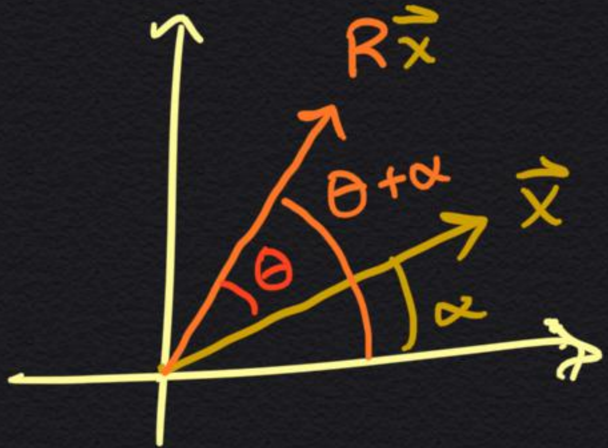
$\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ special case: $\vec{v} = \vec{x}(\alpha=0)$

$$R\vec{v} = \begin{bmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} C_\theta \\ S_\theta \end{bmatrix}$$



$$R\vec{x} = \begin{bmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} C_\alpha \\ S_\alpha \end{bmatrix} = \begin{bmatrix} C_\theta C_\alpha - S_\theta S_\alpha \\ S_\theta C_\alpha + C_\theta S_\alpha \end{bmatrix} = \begin{bmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{bmatrix}$$



Part 2: Commutativity (of matrix operations):

First, find the reflection matrix about the Y-axis:

$$Y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

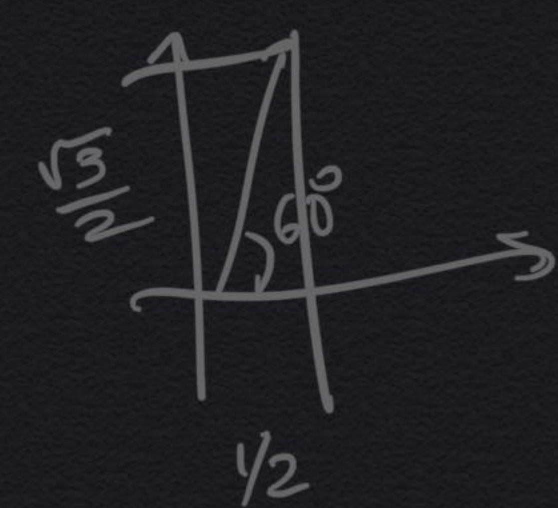
Note: We derived the reflection about the Y-axis by seeing how Y treats $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ← $Y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = +\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$Y\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Doesn't change y-component, but flips x-component.

a) Find the operation for rotating $\vec{x}$ by $\theta = 60°$, then reflecting about the Y-axis:

$$R = \begin{pmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{pmatrix} \Rightarrow R = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$



$YR\vec{x}$

$$M_a = YR = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

→ YR means "rotate, then flip".

b) Find the converse operation (reflect $\vec{x}$ about the Y-axis, then rotate by $\theta = 60°$)

$R Y \vec{x}$

$$M_b = RY = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \checkmark$$

$M_a \neq M_b$

$[YR \neq RY]$

$\checkmark$

$$ABC = A(BC)$$

Associative ✓

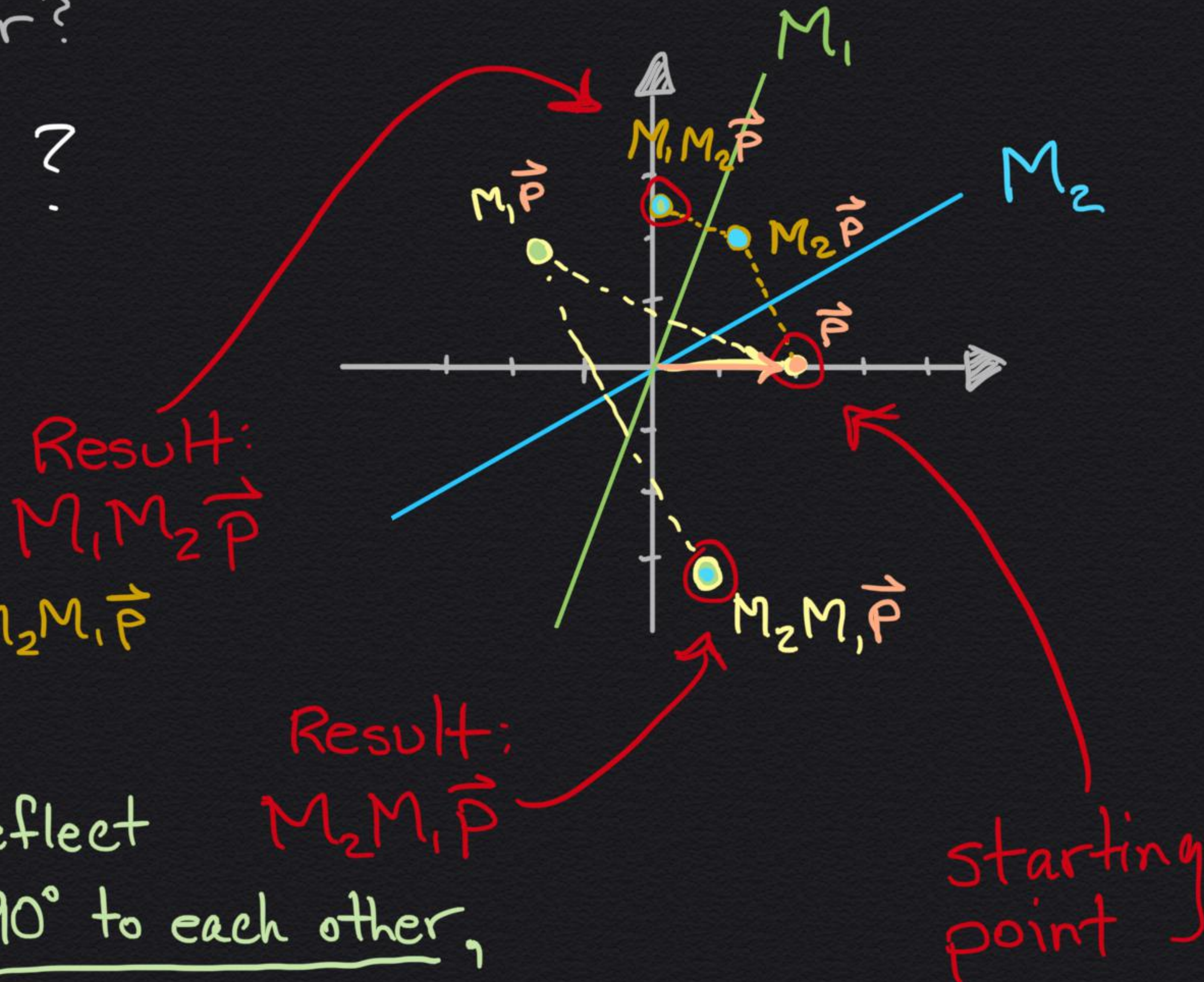
$$= (AB)C$$

c) What do the results from 'a' and 'b' tell you?

✓

d) If you reflected a vector twice (along 2 different axes, do you think the order of those reflections matter?

$$M_1 M_2 \neq M_2 M_1 \;?$$



Yes, it matters!

We see visually $M_1 M_2 \vec{p} \neq M_2 M_1 \vec{p}$
For example vector $\vec{p}$.

Fun fact: if M_1, M_2 reflect about axes that are 90° to each other, then they do commute $M_1 M_2 = M_2 M_1$

Bonus! Show matrix multiplication is distributive

$A^{2\times 2}$

$$A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 \quad ??$$

$$\vec{v}_1 = \begin{bmatrix} v_{1\alpha} \\ v_{1\beta} \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} v_{2\alpha} \\ v_{2\beta} \end{bmatrix}$$

$A(\vec{v}_1 + \vec{v}_2) =$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_{1\alpha} + v_{2\alpha} \\ v_{1\beta} + v_{2\beta} \end{bmatrix} = \begin{bmatrix} a_{11}(v_{1\alpha} + v_{2\alpha}) + a_{12}(v_{1\beta} + v_{2\beta}) \\ a_{21}(v_{1\alpha} + v_{2\alpha}) + a_{22}(v_{1\beta} + v_{2\beta}) \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}v_{1\alpha} + a_{12}v_{1\beta}) + (a_{11}v_{2\alpha} + a_{12}v_{2\beta}) \\ (a_{21}v_{1\alpha} + a_{22}v_{1\beta}) + (a_{21}v_{2\alpha} + a_{22}v_{2\beta}) \end{bmatrix}$$

$$= A\begin{bmatrix} v_{1\alpha} \\ v_{1\beta} \end{bmatrix} + A\begin{bmatrix} v_{2\alpha} \\ v_{2\beta} \end{bmatrix}$$

✓

In Summary:

For matrices A, B, C

Multiplication is...

(1) Associative $A(BC) = (AB)C$

(2) Distributive $A(B+C) = AB + AC$

(3) Not Commutative $AB \neq BA$

Note: It's possible $AB = BA$ for specific matrices A, B (for example 2D rotation matrices like above), but in general it won't hold.