①

$$\overset{1\times2}{A} = \begin{bmatrix} 1 & 4 \end{bmatrix} \qquad \overset{2\times1}{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \overset{2\times2}{C} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad \overset{2\times2}{D} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\overset{2\times4}{E} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \qquad \overset{4\times3}{F} = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \qquad \overset{3\times3}{G} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \qquad \overset{3\times3}{H} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

The gold dimension notation shown here is my way to see which matrices can be multiplied together, and what dimensions the result will have.

Dimensions of result 1×1

Must be equal for valid matrix multiplication

a) $\overset{1\times2}{A}\overset{2\times1}{B} = \begin{bmatrix} 1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot3 + 2\cdot4 \end{bmatrix} = \begin{bmatrix} 3+8 \end{bmatrix} = \begin{bmatrix} 11 \end{bmatrix}$

b) $\overset{2\times2}{C}\overset{2\times2}{D} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot3+4\cdot2 & 1\cdot2+4\cdot1 \\ 2\cdot3+3\cdot2 & 2\cdot2+3\cdot1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 12 & 7 \end{bmatrix}$

$DC \neq CD$

c) $DC = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3\cdot1+2\cdot2 & 3\cdot4+2\cdot3 \\ 2\cdot1+1\cdot2 & 2\cdot4+1\cdot3 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}$

d) $\overset{2\times2}{C}\,\overset{2\times4}{E} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot1+4\cdot4 & 1\cdot9+4\cdot3 & 1\cdot5+4\cdot2 & 1\cdot7+4\cdot2 \\ 2\cdot1+3\cdot4 & 2\cdot9+3\cdot3 & 2\cdot5+3\cdot2 & 2\cdot7+3\cdot2 \end{bmatrix}$

$$= \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}$$

$3 \neq 2$

e) $\overset{4\times3}{F}\,\overset{2\times4}{E} = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}$ Cannot multiply FE

Graphically, why is this?

You can think of $E^{2\times4}$ as a map $\vec{y} = E\vec{x}$ from a 4D space $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ to a 2D space $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

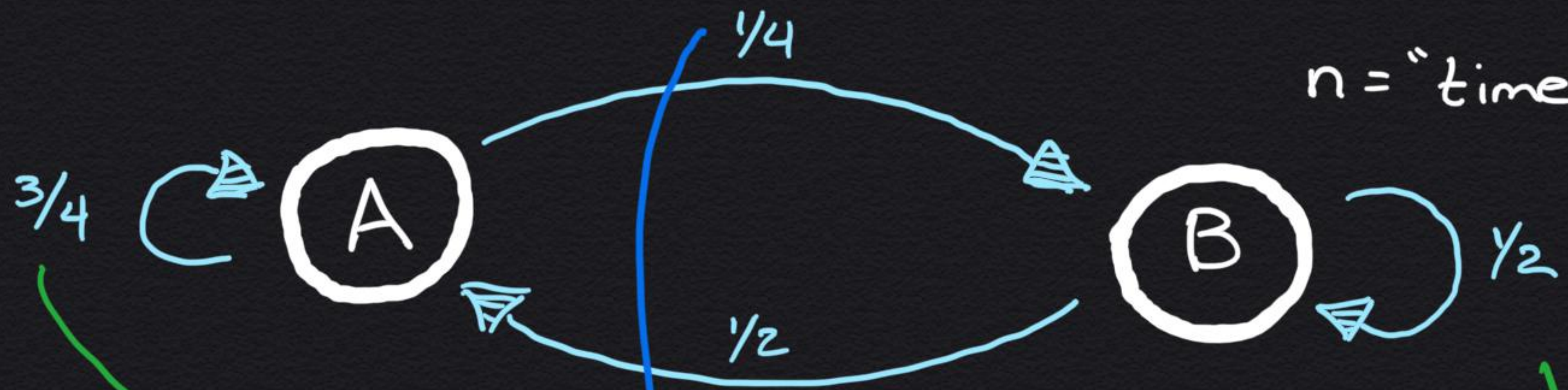
But $F^{4\times3}$ maps from 3D to 4D space. Thus $F\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ does not make any sense!

f) $\overset{2\times4}{E}\,\overset{4\times3}{F} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} (1\cdot5+9\cdot6+5\cdot4+7\cdot3) & (5+9+5+7\cdot2) & (8+9\cdot2+5\cdot7+7\cdot2) \\ \cdots & \cdots & \cdots \end{bmatrix}$$

② Transition Matrix: $\vec{x}[n] = \begin{bmatrix} x_A[n] \\ x_B[n] \end{bmatrix}$

$n =$ "time step"



a) Find the transition matrix such that $\vec{x}[n+1] = S\,\vec{x}[n]$:

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} x_A[n] \\ x_B[n] \end{bmatrix} = \begin{bmatrix} S_{11}\,x_A[n] + S_{12}\,x_B[n] \\ S_{21}\,x_A[n] + S_{22}\,x_B[n] \end{bmatrix} = \begin{bmatrix} x_A[n+1] \\ x_B[n+1] \end{bmatrix}$$

$$\boxed{S = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}}$$

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b] Let's now find the reversed process S^{-1}, which in application takes back a step $\vec{x}[n-1] = S^{-1}\vec{x}[n]$.

"identity matrix I"

Why do we write $\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$ on the right side?

$$S\vec{x} = \vec{y}$$
$$S\vec{x} = I\vec{y}$$
$$\vec{x} = S^{-1}\vec{y}$$

$$\left[\begin{array}{cc|cc} 3/4 & 1/2 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{4}{3}R_1} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right]$$

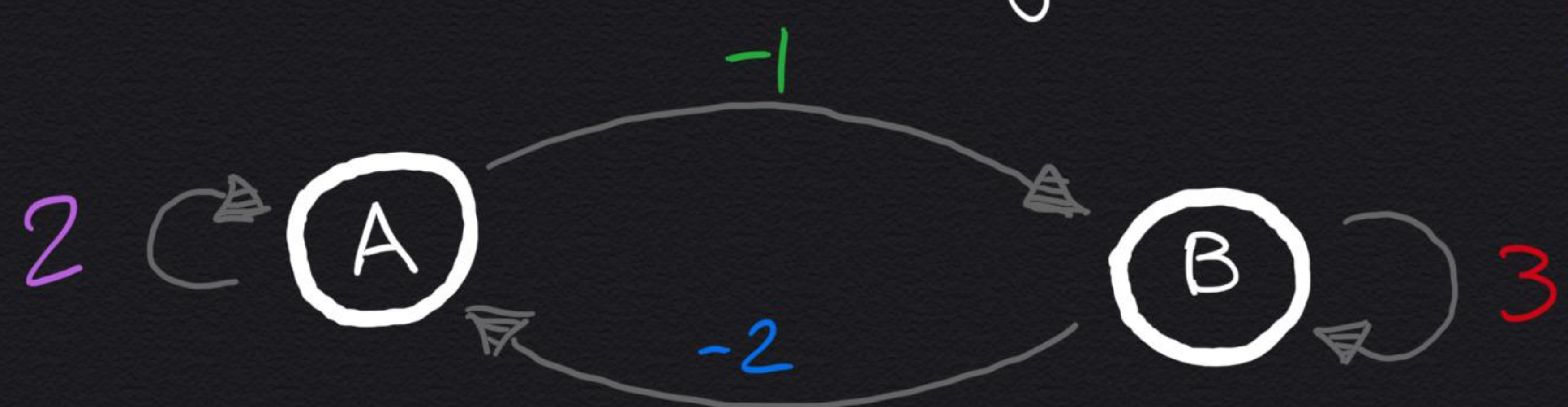
$$\xrightarrow{R_2 \to R_2 - \frac{1}{4}R_1} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & 1/3 & -1/3 & 1 \end{array}\right] \xrightarrow{R_2 \to 3R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - \frac{2}{3}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{3}+\frac{2}{3} & -2 \\ 0 & 1 & -1 & 3 \end{array}\right] = \underbrace{\begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}}_{S^{-1}}$$

$$S^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

We're done once the side of the augmented form becomes an identity.

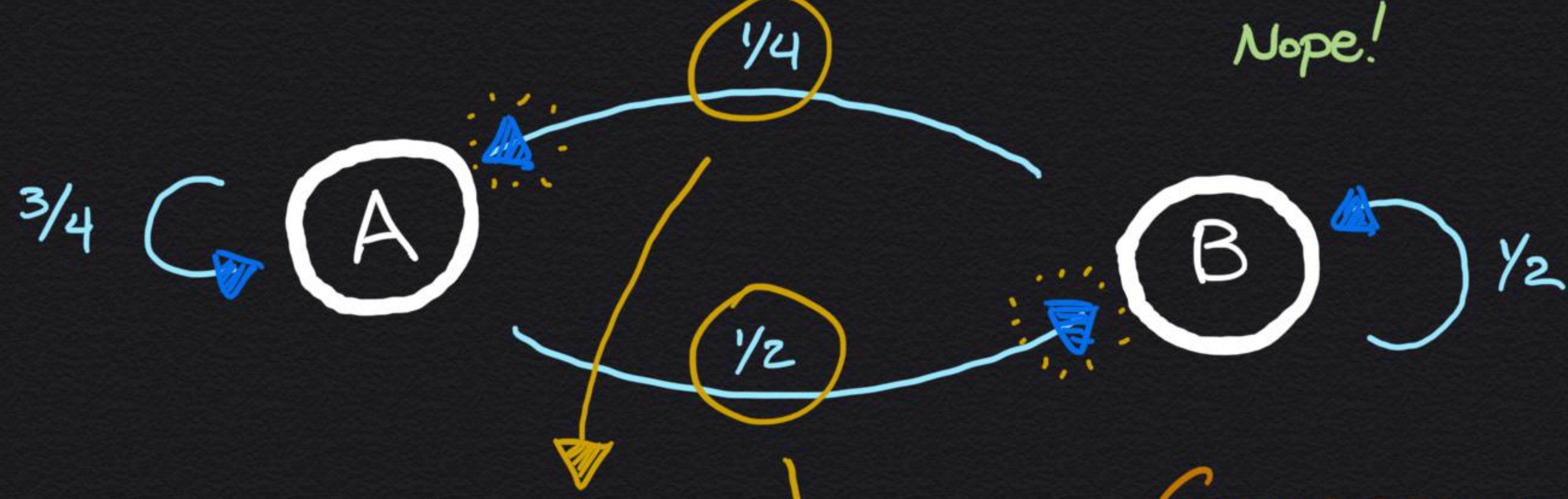
c] Next let's draw out the diagram of S^{-1}:



While we have conservation in S^{-1} (sum of columns are all 1), these flow values are physically nonsensical.

d) Redraw the diagram for S, but with flow directions flipped, and find 'T' transition matrix for the new process. Does $T = S^{-1}$?

Nope!



$$T = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

Compare with the other transition matrices:

$$S = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

Diagonals are not impacted, but the off-diagonals get swapped!

(Fun Fact: whenever we swap the off-diagonals of a matrix, we are taking the "transpose" of the matrix.)

Example: $\underset{\text{"take transpose"}}{A^T} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

Note: We see that $T \neq S^{-1}$. This might seem odd, but notice that an arrow $\xrightarrow{1/4}$ doesn't mean "transfer 1/4 Liters of fluids", it means "transfer 25% of the resevoir over". This ratio idea is key to see why flipping the arrow direction doesn't undo the transfer.

e) Suppose our initial state is $\vec{x}[1] = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$.

Compute the state vector after 2 time-steps $\vec{x}[3]$:

$$\vec{x}[3] = S\vec{x}[2] = S(S\vec{x}[1])$$

$$= (SS)\vec{x}[1]$$

Note! Matrix multiplication is associative, meaning that $(SS)\vec{x} = S(S\vec{x})$. Here we compute this way.

$$= \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \left(\begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 3/4 \cdot 12 + 1/2 \cdot 12 \\ 1/4 \cdot 12 + 1/2 \cdot 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 45/4 + 18/4 \\ 15/4 + 18/4 \end{bmatrix} = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix} = \vec{x}[3]$$

Note! $\vec{x}[2]$ still sums to $12 + 12 = 24$

$\vec{x}[3]$ also sums to 24.
$(63/4 + 33/4 = 96/4 = 24 \checkmark)$