① Identify inverses of the following matrices (if they exist):

Gauss-Jordan Method:

$$[A \mid I] \Leftarrow A\vec{x} = \vec{b} = I\vec{b}$$

GJ

$$[I \mid A^{-1}] \Rightarrow \vec{x} = A^{-1}\vec{b}$$

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{9}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/9 \end{array}\right]$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/9 \end{bmatrix}$$

b) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$

$R_1 \to \frac{1}{a} R_1$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - cR_1$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - c\frac{b}{a} & -c/a & 1 \end{array}\right]$$

Use this substitution (just for simplicity)

$$\left[\Delta = d - c\left(\frac{b}{a}\right)\right]$$

$R_2 \to \frac{1}{\Delta} R_2$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -c\frac{1}{a\Delta} & \frac{1}{\Delta} \end{array}\right]$$

Then pull this term out

$$\Delta a = ad - bc$$

$R_1 \to R_1 - \frac{b}{a} R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + c\frac{b/a}{a\Delta} & -\frac{b/a}{\Delta} \\ 0 & 1 & -c\frac{1}{a\Delta} & \frac{1}{\Delta} \end{array}\right]$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} \Delta + c\frac{b}{a} & -b \\ -c & a \end{bmatrix}$$

$$\Delta + c\frac{b}{a} = d - \cancel{c\frac{b}{a}} + \cancel{c\frac{b}{a}}$$

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Note! 'A' has an inverse only if $ad - bc \neq 0$. This is the condition needed so that the columns stay linearly independent!

This is a little in advance, but this is the definition of something called "the determinant" of A

$$\det(A) = ad - bc$$

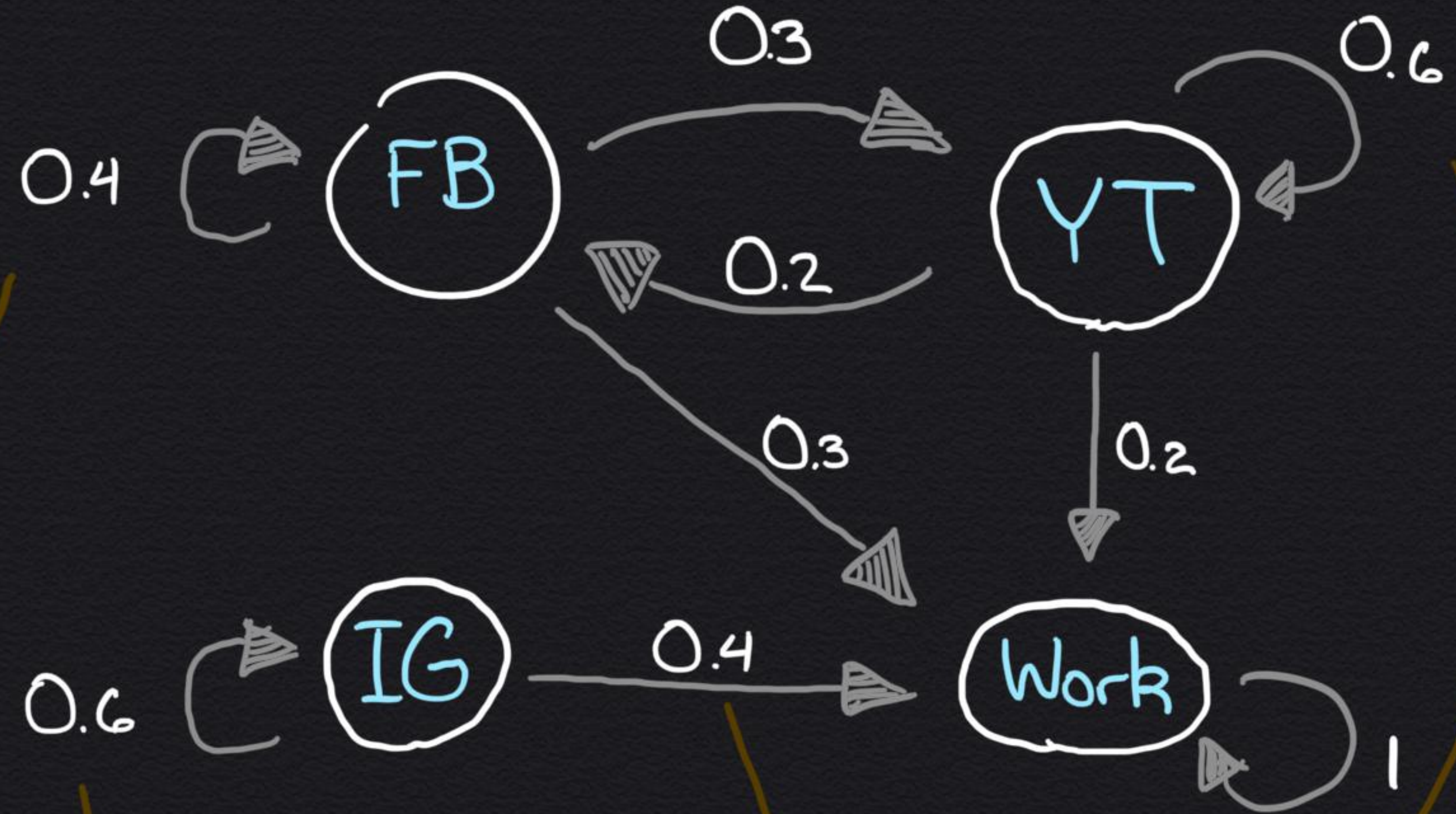
d] $A = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

Note! Since the columns are not linearly independent, A does not have an inverse.

If we went through the work, we'd find that while reducing the matrix the final row becomes all zeros.

It also works the other way; if we can't get the inverse, then the columns are dependent.

② Social Media:



$$\vec{x}[n] \equiv \begin{bmatrix} x_F[n] \\ x_Y[n] \\ x_I[n] \\ x_W[n] \end{bmatrix}$$

a) Derive the transition matrix

$$T = \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}$$

b) Given $\vec{x}[1] = \begin{bmatrix} 700 & 450 & 200 & 150 \end{bmatrix}^T$, find $\vec{x}[2]$:

$$\vec{x}[2] = T\vec{x}[1] = \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} 700 \\ 450 \\ 200 \\ 150 \end{bmatrix} = \begin{bmatrix} 370 \\ 480 \\ 120 \\ 530 \end{bmatrix}$$

c) Compute the sum of each column in the transition matrix:

$$1 \quad 1 \quad 1 \quad 1$$

System is conservative!

d] Could you estimate the steady-state of students from this transition matrix?

$$T\vec{x}_f[n+1] = \vec{x}_f[n]$$

Two methods.

$$\vec{x}_f = \lim_{n \to \infty} T^n \vec{x}[1]$$

$$\vec{x}_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Make this guess (based on the fact that no students leave work once entered).

$$T\vec{x}_f = \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Yay! It's a stead-state solution when all students are working!

Note: In the future we'll build stronger tools to properly analyze these states, yet for now just demonstrating our estimated state will have to be sufficient.