① Column Spaces & Null Spaces

For some matrix $A^{m \times n}$ ...

$$A = \overset{n}{\underset{m}{\begin{bmatrix} \cdots \\ \vdots \end{bmatrix}}}$$

- ...column space = span of column vectors
- ...null space = span of vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{0}$

Given the following matrices, get i. Column space ii. Null space iii. Row-reduced column space iv. Does the column space form a basis?

$$\overset{m \times n}{A\vec{x}} = \begin{bmatrix} \longleftrightarrow \end{bmatrix}\begin{bmatrix} \updownarrow \end{bmatrix} = \vec{y}$$

n Null

m Col

In words, "col(A)" is the space of all $\vec{y}$ you can reach using $\vec{x}$ and multiplying $\vec{y} = A\vec{x}$

$$A\vec{e}_1 = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = \vec{a}_1$$

$$A\vec{x} = \begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$$

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

i. $\text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

ii. $\text{Null}(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

iii. ✓

iv. ✗

Long-form:

$$\left\{ \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$$

$$A\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\alpha = x_2$

One free parameter needed

$x_1 + \cancel{0(\alpha)} = 0 \rightarrow x_1 = 0$

$$\vec{x} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

"Number of needed free parameters = dimension of null(A)"

b) $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

i. col$(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

ii. null$(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

iii. col$(B_{RR}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ ✗

iv. ✗

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + (-1)R_1} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

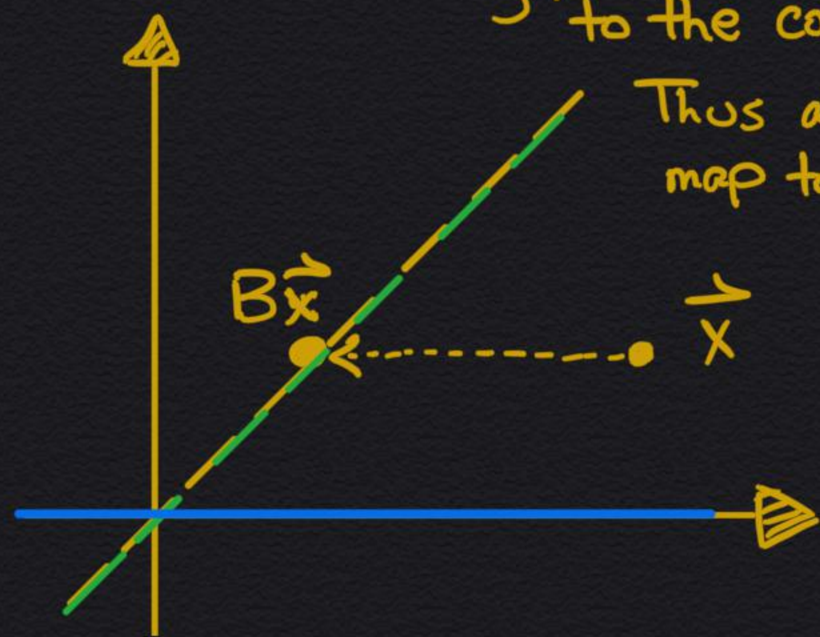
$B\vec{x} = \vec{0}$

$x_1 = \alpha$

$0(\alpha) + 1 \cdot x_2 = 0 \rightarrow x_2 = 0 \qquad \vec{x} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Visually, we see B maps any vector to the col(A) space $\equiv \{x=y\}$.

Thus any $\vec{x}$ with $y = x_2 = 0$ will map to the origin!



$B\vec{x} = B\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix}$

c) $C = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

i. col$(C) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

ii. null$(C) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

iii. col$(C_{RR}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ ✓

iv. Yes, col(C) is basis of $\mathbb{R}^2$

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2/3} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$C\vec{x} = \vec{0} \quad // \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_2 = 0$

$x_1 + 2(\cancel{0}) = 0 \rightarrow x_1 = 0$

$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

d) $D = \begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

i. $col(D) = span\left\{ \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \cancel{\begin{bmatrix} 4 \\ -6 \end{bmatrix}} \right\}$

ii. $null(D) = span\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

iii. $col(D_{RR}) = span\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \cancel{\begin{bmatrix} -2 \\ 0 \end{bmatrix}} \right\}$

iv. No $col(D)$ is <u>NOT</u> a basis for $\mathbb{R}^2$

$D\vec{x} = \vec{0}$

$\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$ $R_1 \to R_1/-2$, $R_2 \to R_2 - 3R_1$

$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$x_2 = \alpha$

$x_1 + (-2)(\alpha) = 0$

$x_1 = 2\alpha \to \vec{x} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix}$

$\vec{x} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

e) $E = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

i. $col(E) = span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \end{bmatrix} \right\}$

ii. $null(E) = span\left\{ \begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

iii. ✓

iv. <u>No</u>, because $col(E)$ has linearly-dependent vectors.

$E\vec{x} = \vec{0}$

$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$ $R_2 \to R_2 - R_1$

$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix}$ $R_2 \to \frac{1}{2}R_2$

$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 5/2 & 1/2 \end{bmatrix}$ $R_1 \to R_1 + R_2$

$\begin{bmatrix} 1 & 0 & 1/2 & -7/2 \\ 0 & 1 & 5/2 & 1/2 \end{bmatrix}$

$x_3 = \alpha \quad x_4 = \beta$

$x_1 + \frac{1}{2}\alpha - \frac{7}{2}\beta = 0$

$x_2 + \frac{5}{2}\alpha + \frac{1}{2}\beta = 0$

$\vec{x} = \begin{bmatrix} -\frac{1}{2}\alpha + \frac{7}{2}\beta \\ -\frac{5}{2}\alpha - \frac{1}{2}\beta \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1/2\alpha \\ -5/2\alpha \\ \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} 7/2\beta \\ -1/2\beta \\ 0 \\ \beta \end{bmatrix}$

$= \alpha \begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$

② Identifying a basis:

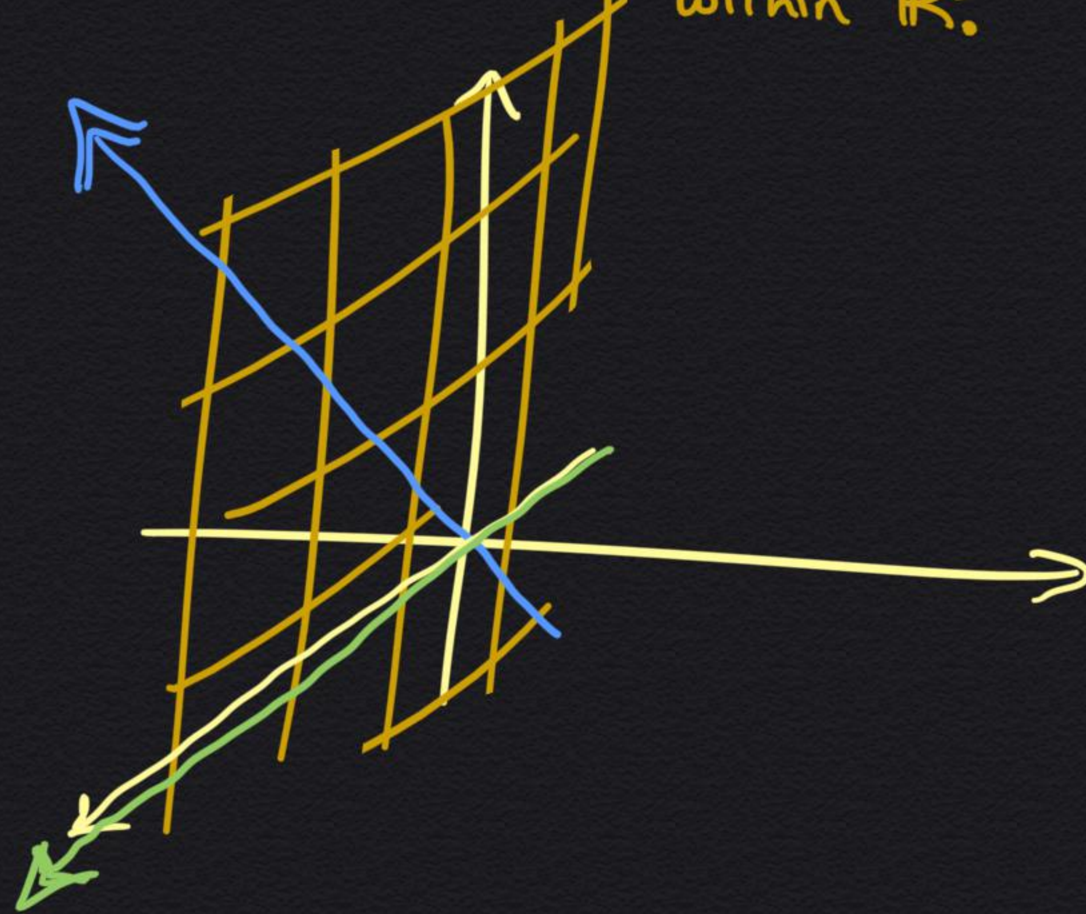
For each set of vectors...

i. Do they describe a basis?

ii. Is the basis for $\mathbb{R}^3$, or different?

a) $V_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Visually, we see that the span form the xz-plane within $\mathbb{R}^3$.



i. No, since this doesn't span $\mathbb{R}^3$

ii. X

b) $V_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to -R_2 \\ R_3 \to -R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

i. Yes! V_2 is a basis of $\mathbb{R}^3$

ii. ✓

Since a) Spans $\mathbb{R}^3$

b) Are linearly independent

Since the row-reduced form of the matrix (formed by the span) shows it has an inverse, the columns must be linearly independent!

c) $V_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

i. No! Not linearly independent.

ii. Not a basis at all.

- By using the machinery above, our row-reduced form here would render a zero row, showing that the span contains linearly dependent vectors.

- Yet, by inspection $\vec{u}_1 - \vec{u}_2 - \vec{u}_3$

$$= \begin{bmatrix} 1-1-0 \\ 1-0-1 \\ 1-1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

proving they're linearly <u>dependent</u>.

- Still $\text{span}\{\vec{u}_1, \vec{u}_2\}$, $\text{span}\{\vec{u}_1, \vec{u}_3\}$, $\text{span}\{\vec{u}_2, \vec{u}_3\}$ are all linearly indepent.

Visually, V_3 is this plane:

$V_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(Plane is coming towards us, sorry for the poor visual :( )