Discussion 5A Notes

Determinant definition for a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

→ " $\det(A) = ad - bc.$ "

Characteristic Polynomial $f(\lambda) = \det(A - \lambda I)$.

Eigenvalues are the roots of this polynomial $f(\lambda) = 0$.

$$A\vec{v} = \lambda\vec{v} \qquad A(\alpha\vec{v}) = \lambda(\alpha\vec{v})$$

Note: $\vec{v}$ is an eigenvector of A, so $A\vec{v}$ is going to be on the same line.
• Yet $\vec{x}$ is not, so $A\vec{x}$ skews off from $\vec{x}$.

Key Properties:

(1) Eigenvalue problems make sense ONLY for square matrices $A^{n\times n}$.

means "if and only if"

(2) For matrix $A^{n\times n}$, $\det(A) = 0$ iff the Null-Space is not empty $A\vec{x} = \vec{0}$

BIG DEAL!!

$$A\vec{v} = \lambda I\vec{v}$$
$$(A - \lambda I)\vec{v} = \vec{0}$$

This is why $\det(A - \lambda I) = 0$ gives us the eigenvalues. But proving (2) is beyond the class's scope.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_2 \to R_2 + sR_1} A' = \begin{pmatrix} a & b \\ c + (sa) & d + (sb) \end{pmatrix}$$

$$\det(A') = a(d+sb) - b(c+sa)$$
$$= ad - bc + \cancel{abs} - \cancel{abs} = \det(A)$$

Notice! This is far from a proof of (2), and if A' is $A \xrightarrow{R_1 \to 2R_1} A'$, then $\det(A') = 2\det(A)$.

Also if A' is $A \xrightarrow{R_1 \rightleftarrows R_2} A'$, then $\det(A') = -\det(A)$.

The key points here: for any Gaussian operations $A \to A'$, then the determinant stays either zero or nonzero

$\det(A') = 0$ if $\det(A) = 0$

$\det(A') \neq 0$ if $\det(A) \neq 0$

① Mechanical Eigenvalues & Eigenvectors

Identify the eigen-pairs for each 2×2 matrix

free-parameter $\alpha \in \mathbb{R}$ is used a lot.

a) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $\quad \det(A - \lambda I) = \det\begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = \lambda(\lambda+3)+2$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda+1)(\lambda+2)$$

$\boxed{\begin{matrix} \lambda = -1 \\ \lambda = -2 \end{matrix}}$

$a\lambda^2 + b\lambda + c = 0$

$$\left[\lambda = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}\right]$$

REMEMBER QUADRATIC FORMULA!!!

...pls ü

$\lambda = -1$ $\quad (A - (-1)I)\vec{v} = \vec{0}$

$0-(-1)$

$-3-(-1)$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right]$

$R_2 \to R_2 + 2R_1$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow v_1 + \alpha = 0$

$\therefore v_1 = -\alpha$

$\alpha = v_2$

$$\boxed{\vec{v}_{\lambda=-1} = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

Note: You can use $\alpha = v_1$ instead, and you'll still get the same solution! (it might be a different scaling though)

$\lambda = -2$: $(A + 2I)\vec{v} = \vec{0}$

$0-(-2) \rightarrow \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$

$R_2 \to R_2 + R_1$

$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \longrightarrow 2v_1 + 1(\alpha) = 0$

$\therefore v_1 = -\frac{1}{2}\alpha$

$\alpha = v_2$

$$\boxed{\vec{v}_{\lambda=-2} = \begin{bmatrix} -\frac{1}{2}\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}$$

b) $B = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$ $\det(B - \lambda I) = \det \begin{bmatrix} -2-\lambda & 4 \\ -4 & 8-\lambda \end{bmatrix} = (\lambda+2)(\lambda-8) + 16$

$= \lambda - 6\lambda - \cancel{16} + \cancel{16}$

$= \lambda(\lambda - 6) \equiv 0$

$\lambda = 0$
$\lambda = 6$

$\lambda = 0$: $\begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$ $B\vec{v} = \vec{0}$

$R_2 \to R_2 - 2R_1$

$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \to -2(\alpha) + 4V_2 = 0$

$V_2 = \frac{1}{2}\alpha \to$

$\lambda = 0$

$\boxed{\vec{V} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} = \frac{\alpha}{2} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

$\alpha = v_1$

An example of setting $v_1 = \alpha$ instead of $v_2 = \alpha$.

Also, I skipped scaling Row 1 (which was lazy of me). But it just meant that I had to scale when I subbed α into the first equation.

c) $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \det(C - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = \lambda^2 \equiv 0$

$(\lambda)(\lambda) = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \boxed{v_2 = 0}$

$\boxed{v_1 = \alpha}$

$$\boxed{\lambda = 0 \qquad \vec{v} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

Note! Because the pivot is on the right ($v_2 = 0$), I can't choose $v_2 = \alpha$ as my free parameter.

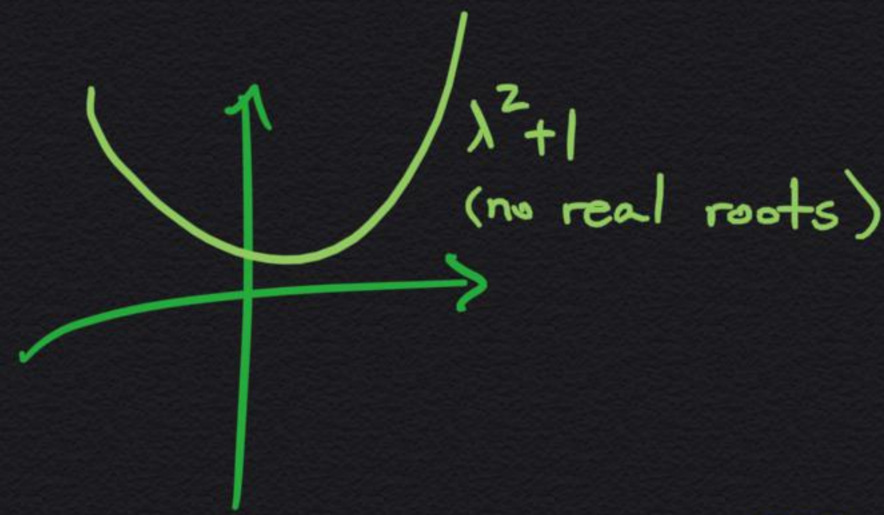
spooky...

Eigenvalues can be complex!!!

Recall...

unit → $(i)^2 = -1$

complex number → $Z = x + iy$



d] $D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 \equiv 0$

$\sqrt{-4} = \sqrt{-1 \cdot 4} = \overset{i}{\sqrt{-1}}\,\overset{2}{\sqrt{4}}$

$$\left[\begin{aligned} \lambda &= -\frac{0}{2} \pm \frac{1}{2}\sqrt{0-4} \\ &= \pm \frac{1}{2} 2i = \pm i \end{aligned}\right]$$

- The quadratic formula works for complex numbers too!!!

$\underline{\underline{\lambda = i}}$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

$R_1 \to iR_1$

$$\begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \to v_1 - i\alpha = 0$$

$\hookrightarrow \underline{\underline{v_1 = i\alpha}}$

$\underline{\underline{\alpha = v_2}}$

$$\Longrightarrow \boxed{\begin{gathered}\lambda = i \\ \vec{v} = \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\end{gathered}}$$

Eigenvectors can also be complex!

(Try not to read heavily into this conceptually just yet pls)

$\lambda = -i$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$$

$$\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}$$

$$\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \to v_1 + i\alpha = 0$$

$v_1 = -i$

$\alpha = v_2$

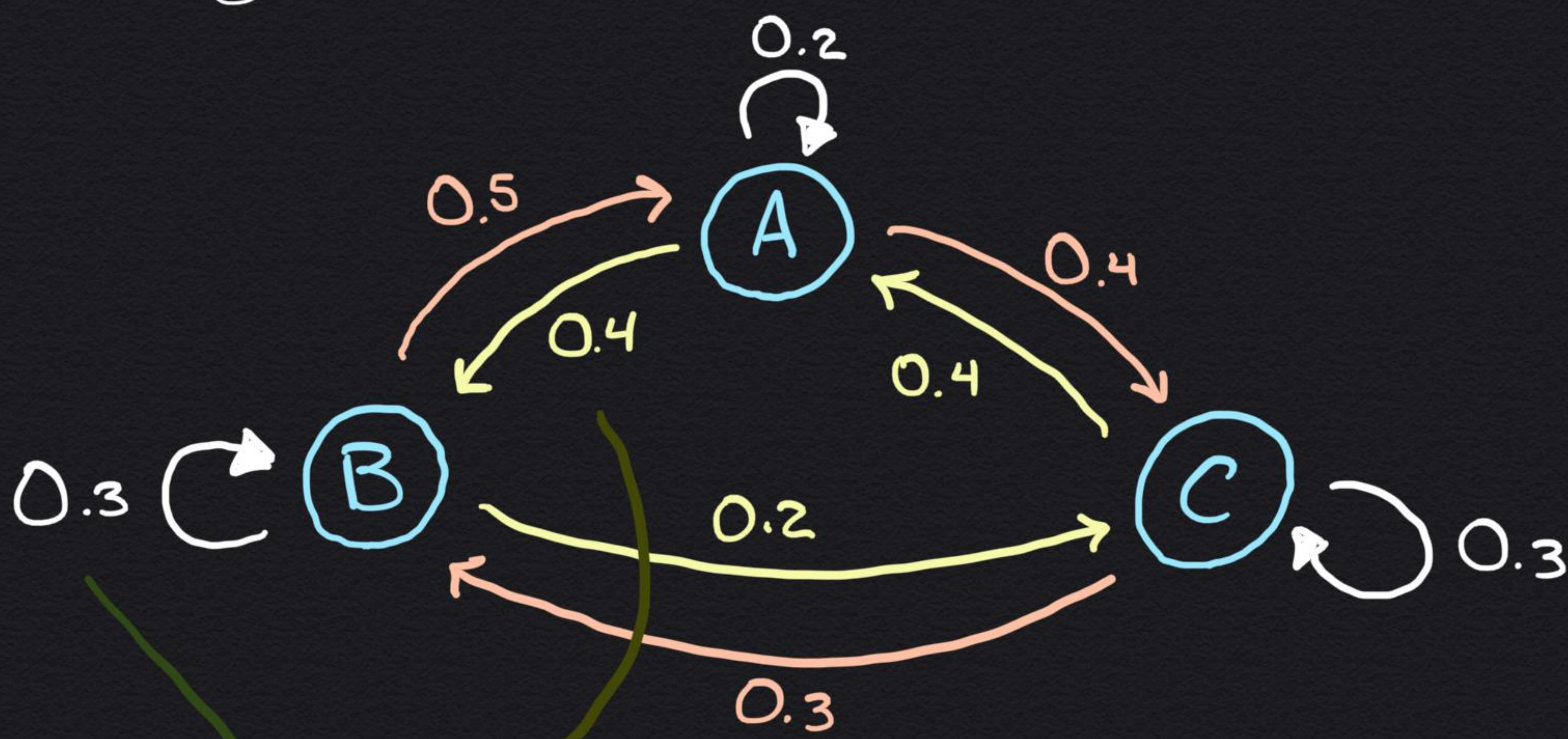
$$\to \boxed{\begin{gathered}\lambda = -i \\ \vec{v} = \alpha\begin{bmatrix} -i \\ 1 \end{bmatrix}\end{gathered}}$$

Fun fact: If every element of a matrix A is real and A has a complex eigenvalue λ, then the conjugate λ^* MUST be another eigenvalue of A.

(In english, if $\lambda = x + iy$ then $\lambda^* = x - iy$)

You can see this for 2x2 matrices thanks to the quadratic formula! But it also holds for $A^{n \times n}$.

② Steady-State Resevoir Levels



a) Write out the transition matrix T, such that $\vec{x}[n+1] = T\vec{x}[n]$

$$T = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

b) Suppose we know that T has the three following eigenvalues

$$\lambda_1 = 1$$

$$\lambda_2 = \frac{-1}{10}(\sqrt{2}+1) \quad -1 < \lambda_2 < 0$$

$$\lambda_3 = \frac{1}{10}(\sqrt{2}-1) \quad 0 < \lambda_3 < 1$$

Try to identi-y the steady-state vector so that $T\vec{x} = \vec{x}$:

$$A\vec{x} - I\vec{x} = \vec{0}$$

$$(A - I)\vec{x} = \vec{0}$$

$0.8 = \frac{4}{5}$

$$\begin{bmatrix} 0.2-1 & 0.5 & 0.4 \\ 0.4 & 0.3-1 & 0.3 \\ 0.4 & 0.2 & 0.3-1 \end{bmatrix}$$

-0.8, -0.7, -0.7

$0.5 \cdot \frac{5}{4} = \frac{5}{8}$

$0.4 \ \frac{5}{4} = \frac{1}{2}$

$R_1 \rightarrow \frac{-5}{4} R_1$

$$\begin{bmatrix} 1 & -5/8 & -1/2 \\ 2/5 & -7/10 & 3/10 \\ 2/5 & 1/5 & -7/10 \end{bmatrix}$$

$R_2 \rightarrow R_2 - \frac{2}{5} R_1$

$$\begin{bmatrix} 1 & -5/8 & -1/2 \\ 0 & \frac{-9}{20} & 1/2 \\ 2/5 & 1/5 & -7/10 \end{bmatrix}$$

$R_3 - \frac{2}{5} R_1$

R_2

$$\begin{bmatrix} 1 & -5/8 & -1/2 \\ 0 & \frac{-9}{20} & 1/2 \\ 2/5 & 1/5 & -7/10 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \frac{2}{5}R_1}$$

$$\begin{bmatrix} 1 & -5/8 & -1/2 \\ 0 & -9/20 & 1/2 \\ 0 & 9/20 & -1/2 \end{bmatrix}$$

$$\frac{1}{5} + \frac{2}{\cancel{5}}\left(\frac{\cancel{5}}{8}\right) = \frac{4}{20} + \frac{5}{20} = \frac{9}{20}$$

$$\frac{-7}{10} + \frac{\cancel{2}}{5}\left(\frac{1}{\cancel{2}}\right) = \frac{-7}{10} + \frac{2}{10} = \frac{-5}{10} = \frac{-1}{2}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \checkmark$$

$$\begin{bmatrix} 1 & -5/8 & -1/2 \\ 0 & -9/20 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \to -\frac{9}{20}V_2 + \frac{1}{2}\alpha = 0$$

$$\left[V_2 = \frac{20}{2 \cdot 9}\alpha = \frac{10}{9}\alpha \right]$$

$$V_3 = \alpha$$

$$V_1 - \frac{5}{8}\left(\frac{10}{9}\alpha\right) - \frac{1}{2}(\alpha) = 0$$

$$V_1 - \frac{25}{36}\alpha - \frac{18}{36}\alpha = 0 \to \left[V_1 = \frac{43}{36}\alpha \right]$$

$$\lambda = 1$$

$$\vec{V} = \begin{bmatrix} \frac{43}{36}\alpha \\ \frac{10}{9}\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 43/36 \\ 10/9 \\ 1 \end{bmatrix}$$

Woof!
Thanks for sticking through ü

A quick chat on eigenvalues in networks:

- The steady-state solution $\vec{x}$ satisfies $T\vec{x} = \vec{x}$, which is the eigenvalue problem with $\lambda = 1$!!!

- Now does a solution always exist? (ie. is $\lambda=1$ always an eigenvalue of T?)

Well, if we suppose that T is conservative (columns sum to one). From HW we saw that this proves that after each transition the state vector has the same sum!

$$\text{SumTerms}(\vec{x}[n]) = \text{SumTerms}(\vec{x}[n+1])$$

where $\vec{x}[n+1] = T\vec{x}[n]$

→ Yet we must satisfy $T\vec{v} = \lambda\vec{v}$ for the eigenvalue problem! So if $\vec{x}[n] = \vec{v}$, then $\text{SumTerm}(\vec{v}) = \lambda \cdot \text{SumTerm}(\vec{v})$

...since $\vec{x}[n+1] = T\vec{v} = \lambda\vec{v}$ and $\text{sum}(\lambda\vec{v}) = \lambda \cdot \text{sum}(\vec{v})$

→ Only 2 possible cases:

1. $\lambda = 1$, hooray! :)
2. $\text{SumTerms}(\vec{v}) = 0$, which would require negative terms and be very non-physical :(

This as far as we can go, but it's still usefull!
If you assume $\lambda=1$ yet find $T - \lambda I$ has an empty null space, then you've shown there is no steady state solution!