1 Steady & Unsteady States

$$M = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = 1, \vec{v}_1$
$\lambda_2 = 2, \vec{v}_2$
$\lambda_3 = \frac{1}{2}, \vec{v}_3$

Since the eigenvalues are known, we can find the eigenvectors by identifying the null-space of $[M - \lambda I]$

Recall: $M\vec{v}_j = \lambda_j \vec{v}_j$

$M(M\vec{x}) = (MM)\vec{x} = M^2\vec{x}$

a) Find the eigenspaces for each λ:

$\{\lambda_1 = 1\}$: $M\vec{v} = \lambda_1 \underset{\uparrow I}{\vec{v}} \longrightarrow \underbrace{(M - \cancel{\lambda_1}^{1} I)}\vec{v} = \vec{0}$

$$\begin{bmatrix} \frac{1}{2}-1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1-1 & -2 \\ 0 & 0 & 2-1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to -2R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -\frac{1}{2}R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to v_3 = 0$$

Must choose v_1 or v_2 to have free parameter

$v_2 = \alpha$

$v_1 - (\alpha) + \cancel{0} = 0 \to v_1 = \alpha$

$$\vec{v} = \begin{bmatrix} \alpha \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_1 = 1 \quad \vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\{\lambda_2 = 2\}$

$$\underbrace{(M-\cancel{\lambda_2}^{\,2} I)}\vec{v}=\vec{0}$$

- Verify solution:

$$\underset{M}{\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}} \underset{\vec{V}_2}{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}} = \begin{bmatrix} -\frac{1}{2}-1-\frac{1}{2} \\ -2-2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} = \underset{\lambda_2 \vec{V}_2}{2\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} -3/2 & 1/2 & -1/2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to -\frac{2}{3}R_1$

$R_2 \to -R_2$

$$\begin{bmatrix} \underline{1} & -1/3 & +1/3 \\ 0 & \underline{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\to V_1 - \frac{1}{3}(-2\alpha) + \frac{1}{3}(\alpha) = 0 \to V_1 = -\alpha$

$\to V_2 + 2(\alpha) = 0 \to V_2 = -2\alpha$

$V_3 = \alpha$

$$\lambda_2 = 2 \qquad \vec{V}_2 = \alpha\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$\{\lambda_3 = \frac{1}{2}\}$

$$\underbrace{(M-\tfrac{1}{2}I)}\vec{V} = 0$$

$$\begin{bmatrix} 0 & 1/2 & -1/2 \\ 0 & 1/2 & -2 \\ 0 & 0 & 3/2 \end{bmatrix}$$

$R_1 \to 2R_1$

$R_2 \to 2R_2$

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -\frac{1}{3}R_2$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 0 & \underline{1} & -1 \\ 0 & 0 & \underline{1} \\ 0 & 0 & 0 \end{bmatrix}$$

$\to V_2 - 0 = 0 \to \underline{V_2 = 0}$

$\to V_3 = 0$

$V_1 = \alpha$

$$\lambda_3 = \tfrac{1}{2} \qquad \vec{V}_3 = \alpha\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

b) Suppose our intial state is

$$\vec{x} = \vec{x}[1] = \alpha \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3 .$$

For each case, can you determine if the state converges? And if so, to which $\vec{x}_f = \lim_{n\to\infty} M^n \vec{x}[1]$?

$$\vec{x}[2] = M\vec{x} = M(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3)$$
$$= \alpha(M\vec{v}_1) + \beta(M\vec{v}_2) + \gamma(M\vec{v}_3)$$
$$= \alpha(1\,\vec{v}_1) + \beta(2\,\vec{v}_2) + \gamma(\tfrac{1}{2}\vec{v}_3)$$
$$= (\alpha)\vec{v}_1 + (2\beta)\vec{v}_2 + (\tfrac{1}{2}\gamma)\vec{v}_3$$

$\lambda_1 = 1$

$\lambda_2 = 2$

$\lambda_3 = \frac{1}{2}$

$$\vec{x}[3] = M^2\vec{x} = M(M\vec{x}) = \alpha(\overset{1}{\cancel{M}}\vec{v}_1) + 2\beta(\overset{2}{\cancel{M}}\vec{v}_2) + \tfrac{1}{2}\gamma(\overset{1/2}{\cancel{M}}\vec{v}_3)$$
$$= \alpha\vec{v}_1 + (2)^2\beta\vec{v}_2 + (1/2)^2\gamma\vec{v}_3$$

$$\left\{ \vec{x}[n+1] = M^n\vec{x}[1] = \qquad \right\}$$

α	β	γ	$\vec{x}_f$
0	0	$\neq 0$	$\vec{x}_f = 0$
0	$\neq 0$	0	Divergent
0	$\neq 0$	$\neq 0$	Divergent
$\neq 0$	0	0	$\vec{x}_f = \alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	$\vec{x}_f = \alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	Divergent
$\neq 0$	$\neq 0$	$\neq 0$	Divergent

$$\vec{x}_f = \lim_{n\to\infty} M^n\vec{x}[1]$$

$$\alpha\vec{v}_1 + \overset{\infty\text{, Diverges!!}}{\cancel{(2)^n}}\beta\vec{v}_2 + \overset{0}{\cancel{(1/2)^n}}\gamma\vec{v}_3$$

2 Eigenvalues & Special Matrices

a] Does the identity matrix $\mathbb{I}^{n\times n}$ have any eigenvalues, and eigenvectors? $\vec{x} \in \mathbb{R}^n$

$$\mathbb{I}\vec{x} = \vec{x}$$

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

$\underline{\lambda_1 = 1}$

$$\vec{v}_1 \in \{\mathbb{R}^n\} \equiv \left\{ \vec{v} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

- By definition, every $\vec{x} \in \mathbb{R}^n$ is an eigenvector of $\mathbb{I}$ with eigenvalue $\lambda = 1$. Mechanically it is easiest to see this using elementary vectors:

$$\vec{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} 1 \\ 2 \\ \vdots \\ j-1 \\ j \\ j+1 \\ \vdots \\ n \end{array}$$

(so only the j^{th} component is 1, otherwise 0)

For $j = 1, 2, \ldots, n$

- This a good example that shows how eigenvalues may correspond to an eigenspace that is spanned by multiple (linearly independent) vectors.

b) How about the diagonal matrix?

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & d_n \end{bmatrix}$$

The eigenvalues are d_j for $j = 1, 2, \ldots, n$ corresponding the elementary vectors $\vec{e}_j$

$$D\vec{e}_1 = \begin{bmatrix} d_1 & 0 \cdots & 0 \\ 0 & d_2 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 \cdots & d_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} d_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 \vec{e}_1 \qquad \boxed{\lambda_1 = d_1, \quad \vec{v}_1 = \vec{e}_1}$$

$$D\vec{e}_2 = \begin{bmatrix} 0 \\ d_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_2 \vec{e}_2 \qquad \boxed{\lambda_2 = d_2, \quad \vec{v}_2 = \vec{e}_2}$$

$$\boxed{D\vec{e}_j = d_j \vec{e}_j}$$

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1 \cdot 1 + \cancel{d_2 \cdot 0} \\ \cancel{0} \cdot 1 + \cancel{d_2 \cdot 0} \end{bmatrix} = \begin{bmatrix} d_1 \\ 0 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

2x2 Example:

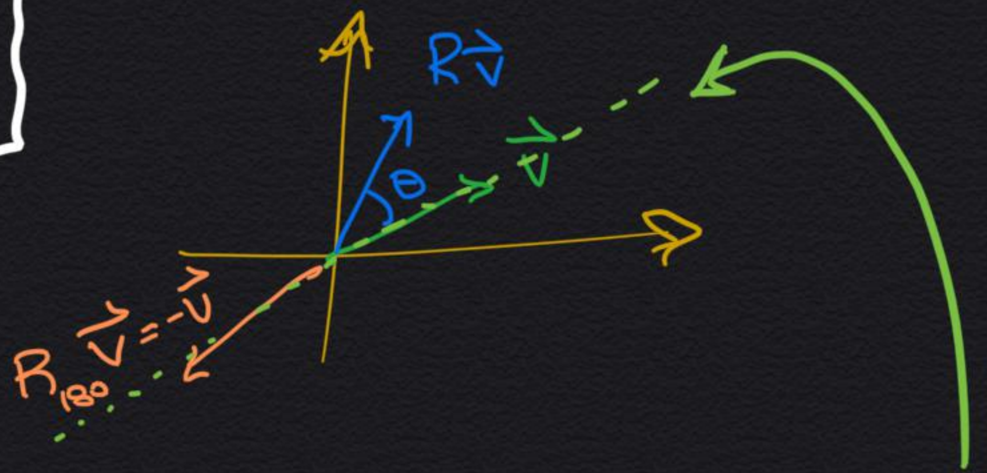
$$(D - d_2 I)\vec{v} = \vec{0}$$

$$\hookrightarrow \begin{bmatrix} d_1 - d_2 & 0 \\ 0 & \cancel{d_2 - d_2}\, 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \boxed{\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \alpha}$$

$v_1 = 0 \qquad \alpha = v_2$

c) How about the rotation matrix of $\mathbb{R}^2$?

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$



$\theta = 0^\circ$ ✓ $\rightarrow \lambda = 1$

180° ✓ $\rightarrow \lambda = -1$

$$R_{180} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The only way that $R\vec{v}$ maps back into the span of $\vec{v}$ is for $\theta = 0^\circ$ (identity) or $\theta = 180^\circ$ (inversion).

In these cases, the eigenspace is all of $\mathbb{R}^2$.

d) Can you compute the eigenvalues for R?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

$$R = \begin{bmatrix} \cos(\theta)' & -\sin(\theta)' \\ \sin(\theta)' & \cos(\theta)' \end{bmatrix}$$

$\cos(\theta) - \lambda$ $\qquad$ $\cos(\theta) - \lambda$

$$\det(R - \lambda I) = (\cos(\theta) - \lambda)^2 + \sin^2(\theta)$$

$$= \lambda^2 - 2\cos(\theta)\lambda + 1 \equiv 0$$

$$\lambda = -\frac{\cancel{2}b}{\cancel{2}a} \pm \frac{1}{\cancel{2}a}\sqrt{\cancel{4}b^2 - \cancel{4}ac}$$

$$\cancel{a}\lambda^2 + 2b\lambda + c = 0$$

$$= -b \pm \sqrt{b^2 - c}$$

$$\lambda = \cos(\theta) \pm \sqrt{\underbrace{\cos^2(\theta) - 1}_{-\sin^2(\theta)}}$$

$$\sqrt{-1}^{\,i}\sqrt{\sin^2(\theta)}$$

$$\lambda_{\pm} = \cos(\theta) \pm i\sin(\theta)$$

Only real when $\theta = 0°$ or $180°$

Eigenvectors: $(R - \lambda_{\pm} I)$

$$\begin{bmatrix} \cancel{\cos(\theta)} - \cancel{\cos(\theta)} \mp i\sin(\theta) & -\sin(\theta) \\ \sin(\theta) & \cancel{\cos(\theta)} - \cancel{\cos(\theta)} \mp i\sin(\theta) \end{bmatrix}$$

$$\sin(\theta)\begin{bmatrix} \mp i & -1 \\ 1 & \mp i \end{bmatrix}$$

$0°$ or $180°$ $\rightsquigarrow$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\vec{V}_{\pm} = \begin{bmatrix} \pm i \\ 1 \end{bmatrix}$$

$$\frac{1}{i} \equiv -i$$

Why is that?

$$i\left(\frac{1}{i}\right) = 1 \equiv i(-i) \checkmark$$

$$= i^2(-1) = 1$$

otherwise...

$$R_1 \to R_1/\mp i$$

$$\begin{bmatrix} \cancel{\mp i}^{\,1}/\mp i & -1/\mp i \\ 1 & \mp i \end{bmatrix} = \begin{bmatrix} 1 & \mp i \\ 1 & \mp i \end{bmatrix} \to \begin{bmatrix} 1 & \mp i \\ 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2 - R_1$$

$V_1 = \pm i\alpha$

$V_2 = \alpha$

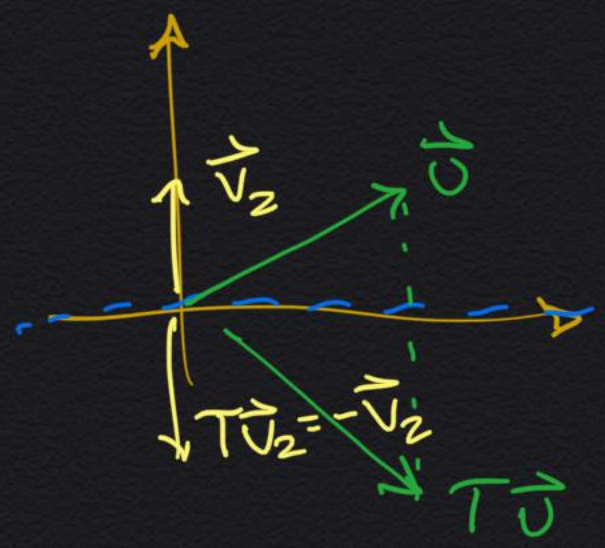
e] How about the reflection matrix T of $\mathbb{R}^2$ about the x-axis? $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(How about 2D reflection matrices in general?)

It's a diagonal matrix, so we can exploit (b):

- $\lambda_1 = 1 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ on x-axis $\Rightarrow T\vec{v}_1 = \vec{v}_1$

  $\lambda_2 = -1 \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 90° to x-axis $\Rightarrow T\vec{v}_2 = -\vec{v}_2$



- For general 2x2 reflection matrices $\lambda = \pm 1$ still, but the eigenvectors will instead be on & 90° to the new reflection axis.

f] Suppose $M^{n\times n}$ has an eigenvalue $\lambda=0$.
What can you say about M's null space?

(can you say anything about solutions to $M\vec{x}=\vec{b}$?)

$$M\vec{v} = \cancel{\lambda}^{0}\vec{v} = \vec{0}$$

Null Space of M is nonempty!!!

There are either no solutions, or infinite (∞) solutions

→ Suppose we have an $\vec{x}$ s.t. $M\vec{x} = \vec{b}$

$$M(\vec{x}+\alpha\vec{v}) = \cancel{M\vec{x}}^{\vec{b}} + \alpha \cancel{M\vec{v}}^{\alpha\cancel{\lambda}^{0}\vec{v}} = \vec{b}$$

$\vec{x}+\alpha\vec{v}$ is Also a solution

→ Otherwise $M\vec{x}=\vec{b}$ has no solutions

G.E{M} ⇒ A zero row

$$\left[\begin{array}{ccccc|c} \ddots & & & & & \\ & & & & & \text{Example} \\ 0 & 0 & 0 & 0 & 0 & b_1 - 2b_2 + b_3 - \dots \end{array}\right]$$

After Gaussian elimination, there will be at least 1 row of zeros on the left (non-empty null).

If the right side is nonzero, then there is no solution!