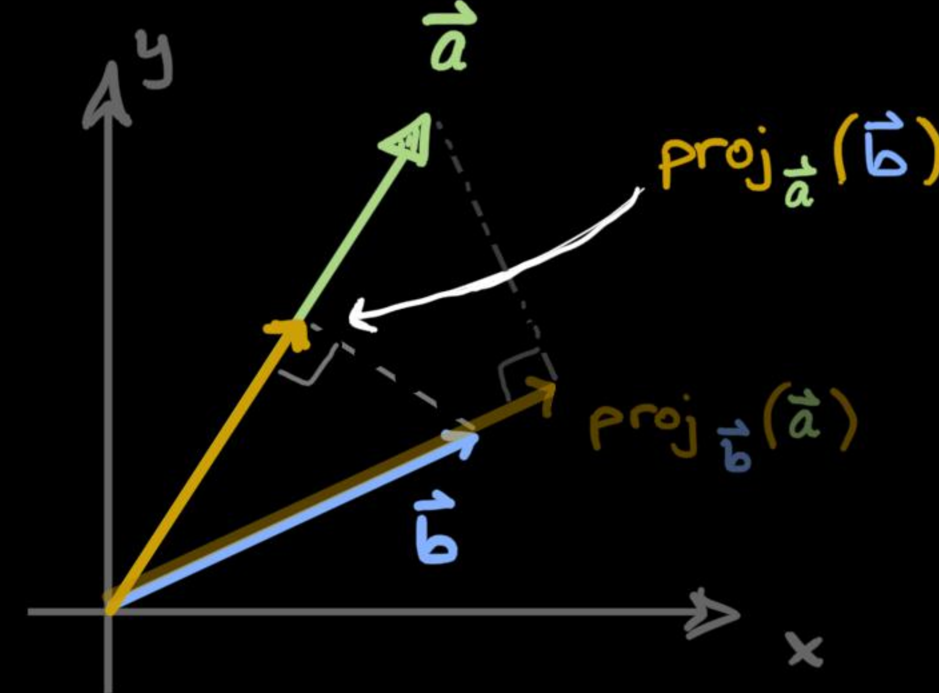
① Mechanical Projection

We define a projection

$$\text{proj}_{\vec{a}}(\vec{b}) = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\|^2} \vec{a}$$

"project vector $\vec{b}$ ONTO vector $\vec{a}$"

*recall $\|\vec{a}\|^2 = \langle \vec{a}, \vec{a} \rangle$.



Compute projections of the following pairs:

a) onto $\vec{a}$

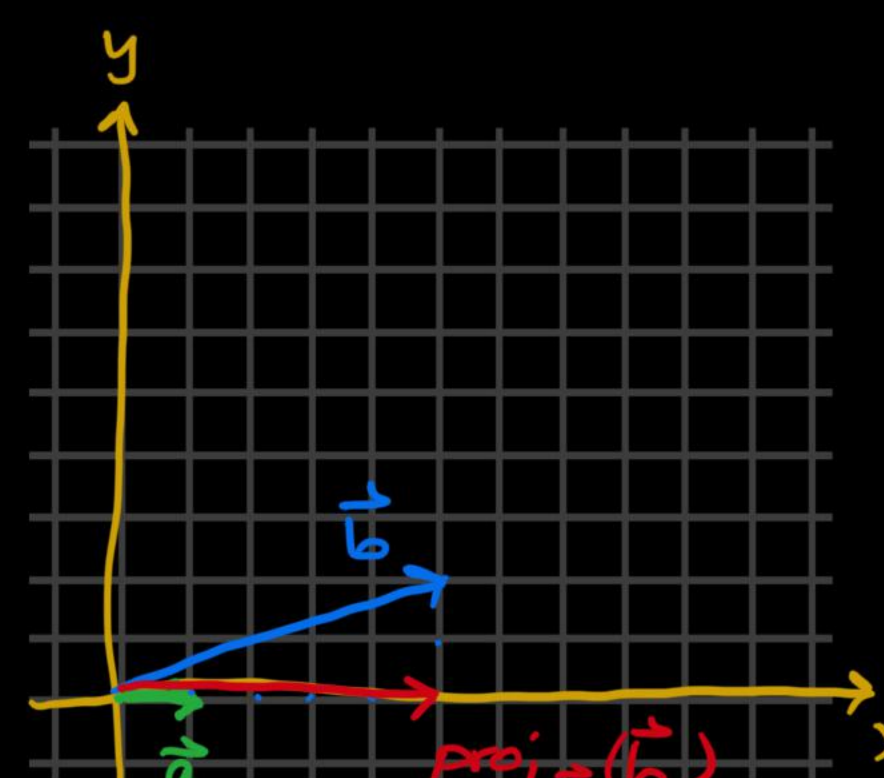
$\text{proj}_{\vec{a}}(\vec{b})$: $\vec{a} = \begin{bmatrix} 1 \; 2 \\ 0 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\langle \vec{a}, \vec{b} \rangle = 1_2 \cdot 5 + 0 \cdot 2 = 5_{10}$

$\|\vec{a}\|^2 = 1_2 \cdot 1_2 + 0 \cdot 0 = 1_4$

$\text{proj}_{\vec{a}}(\vec{b}) = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$\frac{10}{4}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ ✓



Result is independent of $\vec{a}$'s length! Try $\vec{a} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

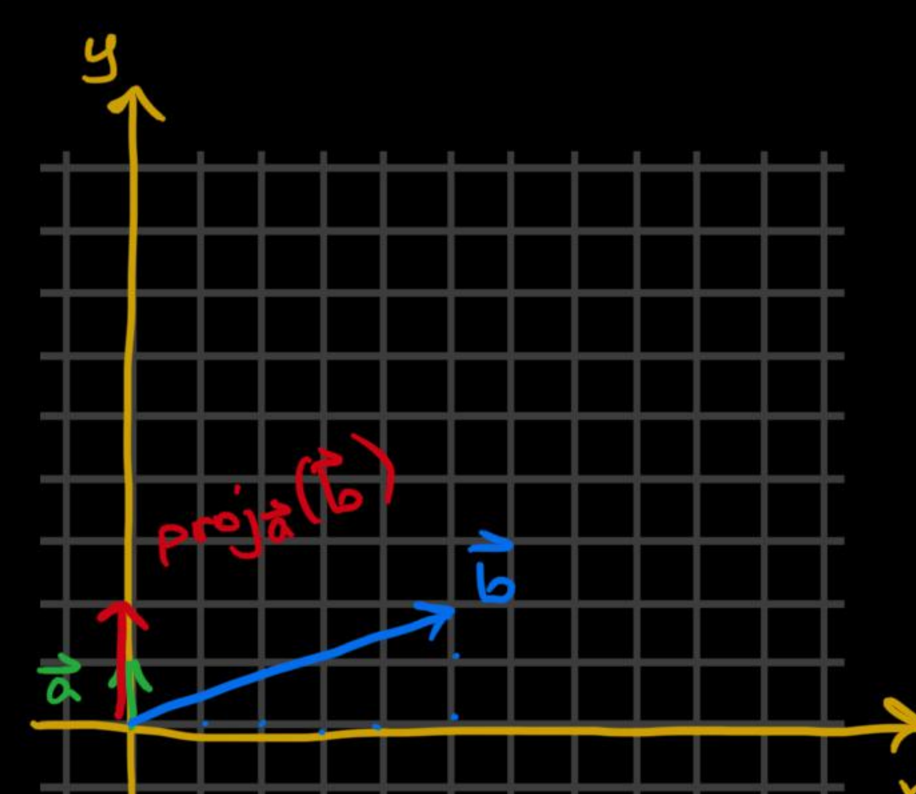
b) onto $\vec{a}$

$\text{proj}_{\vec{a}}(\vec{b})$: $\vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\langle \vec{a}, \vec{b} \rangle = 2$

$\|\vec{a}\|^2 = 1$

$\text{proj}_{\vec{a}}(\vec{b}) = \frac{2}{1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

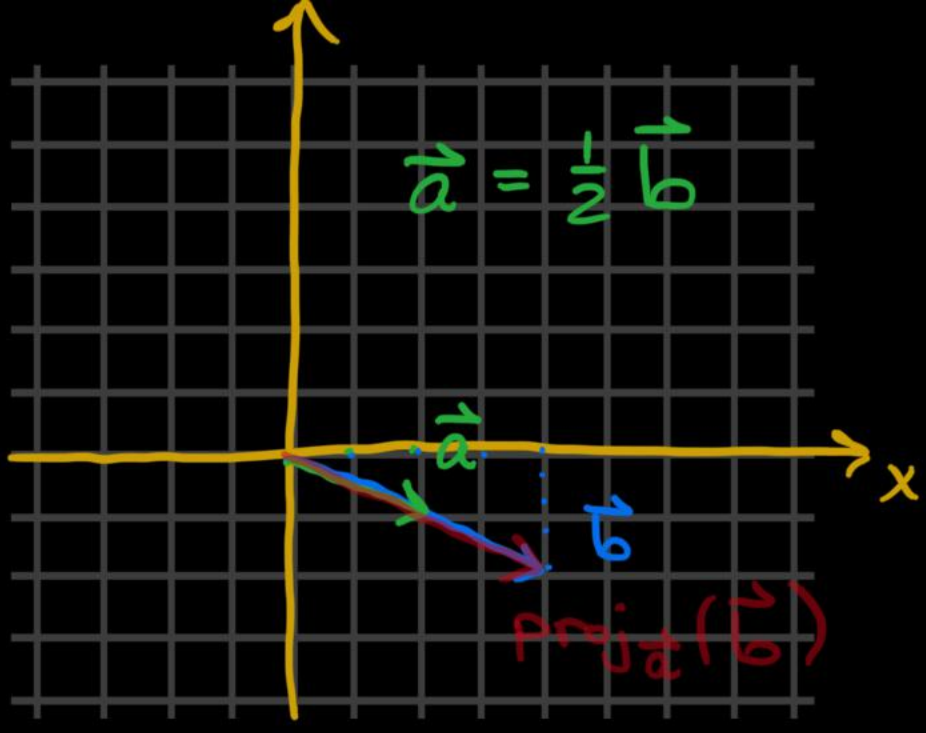


c) $\text{proj}_{\vec{a}}(\vec{b})$: $\vec{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

onto $\vec{a}$

$$\langle \vec{a}, \vec{b} \rangle = 2 \cdot 4 + (-1)(-2) = 10$$
$$\|\vec{a}\|^2 = 4 + 1 = 5$$

$$\text{proj}_{\vec{a}}(\vec{b}) = \frac{10}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

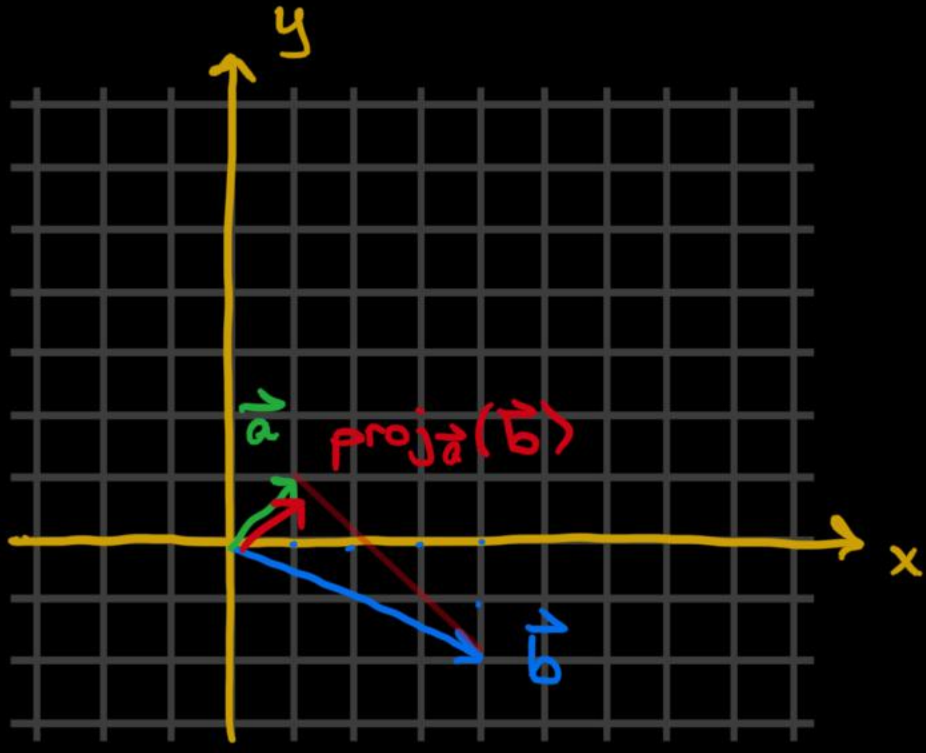


d) $\text{proj}_{\vec{a}}(\vec{b})$: $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

onto $\vec{a}$

$$\langle \vec{a}, \vec{b} \rangle = 1 \cdot 4 + 1 \cdot (-2) = 2$$
$$\|\vec{a}\|^2 = 1^2 + 1^2 = 2$$

$$\text{proj}_{\vec{a}}(b) = \frac{2}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



e) $\text{proj}_A(\vec{b})$: $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

onto $\text{col}(A) = \{\vec{a}_1, \vec{a}_2\}$

This is a bit different: $\text{Proj}_A(\vec{b}) = A(A^TA)^{-1}A^T\vec{b}$

$$A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

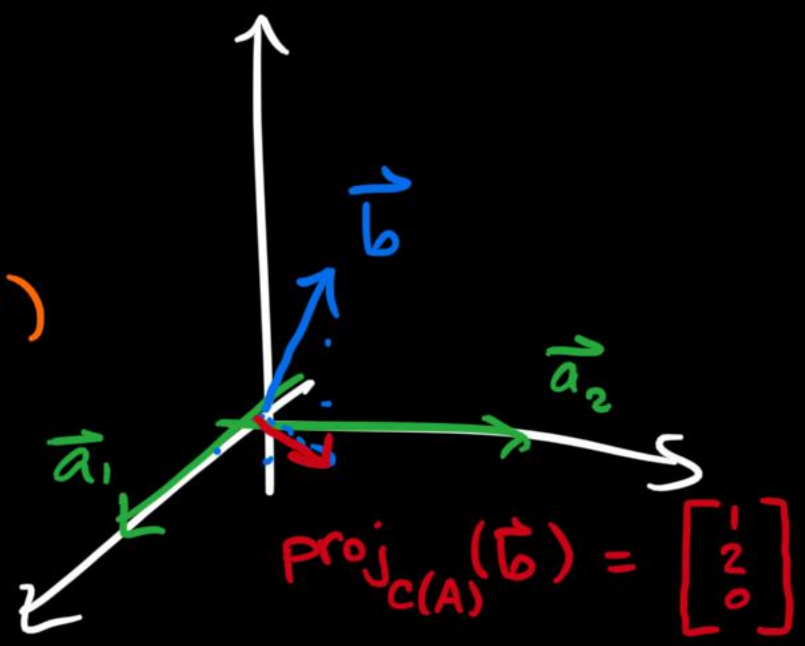
Creates identity!! (Inverse of I is I)

$$(A^TA)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$AIA^T = AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

No z-part!



Thus...

$$A(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

② Least squares; orthogonal columns

We're trying to solve $A\vec{x} = \vec{b}$ for $\vec{x}$, but its impossible given our $A^{3\times 2}$ matrix and $\vec{b} \in \mathbb{R}^3$.

So we then want to minimize $\vec{e} = \vec{b} - A\vec{x}$, formally:

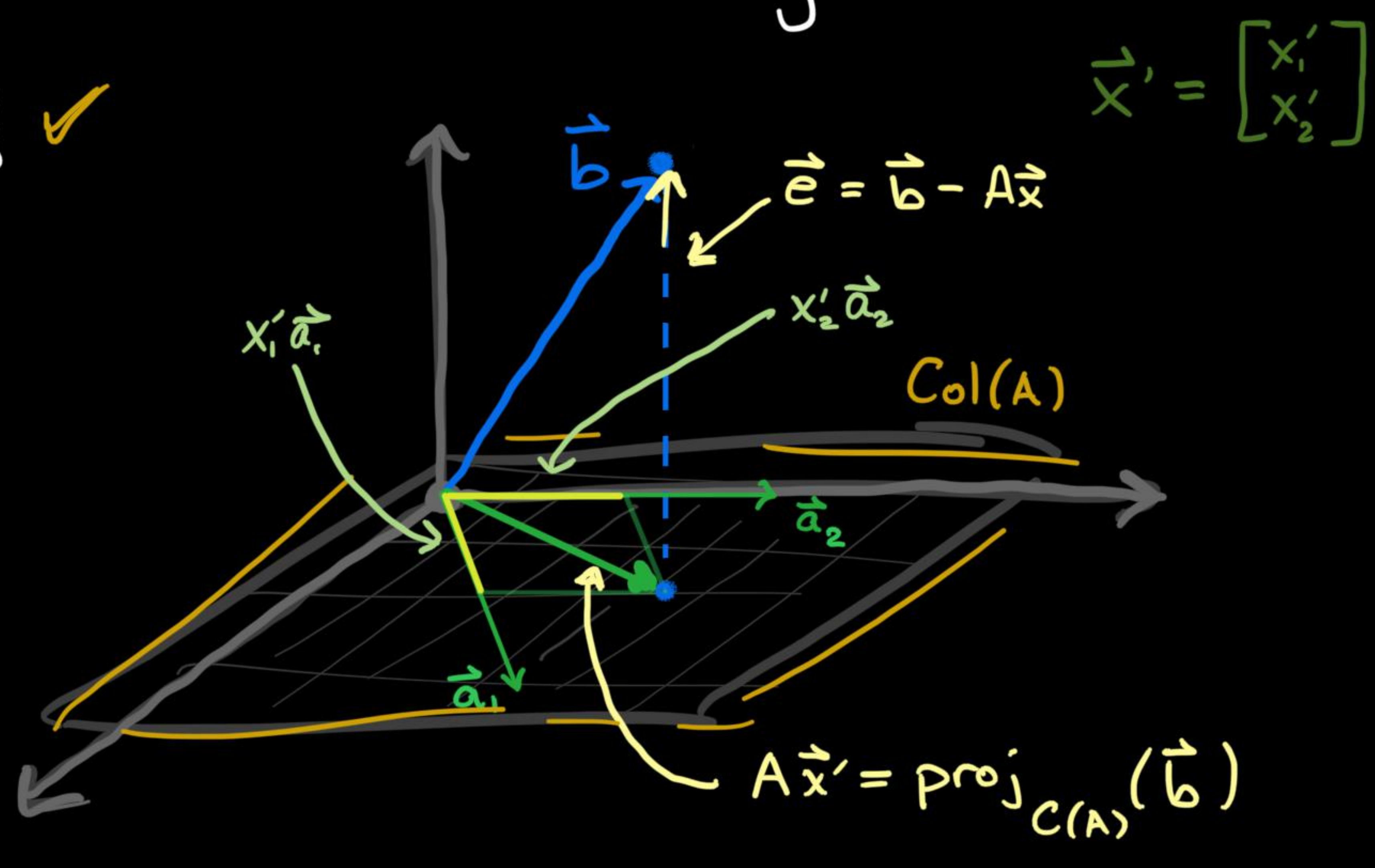
$$\left\{ \text{Find } \underline{\underline{\vec{x}'}} \in \mathbb{R}^2 \text{ so } \|\vec{b} - A\vec{x}'\|^2 \leq \|\vec{b} - A\vec{x}\|^2 \text{ for any } \vec{x} \in \mathbb{R}^2. \right\}$$

We use the formula: $\boxed{\vec{x}' = (A^T A)^{-1} A^T \vec{b}}$

↳ but why does it work???

a) On the diagram, label the following:

* $\text{span}\{\vec{a}_1, \vec{a}_2\}$ ✓
* $A\vec{x}$ ✓
* $x_1' \vec{a}_1$ ✓
* $x_2' \vec{a}_2$ ✓
* $\text{Col}(A)$ ✓
* $\vec{e} = \vec{b} - A\vec{x}'$ ✓
* $\text{proj}_{C(A)}(\vec{b})$ ✓



b) For (orthoganal) columns in A, (so $\langle \vec{a}_1, \vec{a}_2 \rangle = 0$)
show that (from $\vec{x}' = (A^TA)^{-1}A^T\vec{b}$)

$$\vec{x}' = \begin{bmatrix} \dfrac{\langle \vec{a}_1, \vec{b} \rangle}{\|\vec{a}_1\|^2} \\ \dfrac{\langle \vec{a}_2, \vec{b} \rangle}{\|\vec{a}_2\|^2} \end{bmatrix} \quad \& \quad \text{proj}_{C(A)}(\vec{b}) = \text{proj}_{\vec{a}_1}(\vec{b}) + \text{proj}_{\vec{a}_2}(\vec{b})$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\vec{x}' = \underbrace{(A^TA)^{-1}}A^T\vec{b}$$

due to orthogonality!!

$$\begin{bmatrix} \leftarrow \vec{a}_1^T \rightarrow \\ \leftarrow \vec{a}_2^T \rightarrow \end{bmatrix}\begin{bmatrix} \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \langle \vec{a}_1, \vec{a}_1 \rangle & \cancel{\langle \vec{a}_1, \vec{a}_2 \rangle} \\ \cancel{\langle \vec{a}_2, \vec{a}_1 \rangle} & \langle \vec{a}_2, \vec{a}_2 \rangle \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad} & 0 \\ 0 & \frac{a}{ad} \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\|\vec{a}_1\|^2} & 0 \\ 0 & \frac{1}{\|\vec{a}_2\|^2} \end{bmatrix}$$

Recall...

$$\|\vec{a}\|^2 = a[1]^2 + a[2]^2 + \cdots a[n]^2 = a[1]\cdot a[1] + \cdots + a[n]\cdot a[n]$$
$$= \langle \vec{a}, \vec{a} \rangle$$

$$\vec{x}' = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} \frac{1}{\|\vec{a}_1\|^2} & 0 \\ 0 & \frac{1}{\|\vec{a}_2\|^2} \end{bmatrix}\begin{bmatrix} \leftarrow \vec{a}_1^T \rightarrow \\ \leftarrow \vec{a}_2^T \rightarrow \end{bmatrix}\begin{bmatrix} \uparrow \\ \vec{b} \\ \downarrow \end{bmatrix} = \begin{bmatrix} \frac{1}{\|\vec{a}_1\|^2} & 0 \\ 0 & \frac{1}{\|\vec{a}_2\|^2} \end{bmatrix}\begin{bmatrix} \langle \vec{a}_1, \vec{b} \rangle \\ \langle \vec{a}_2, \vec{b} \rangle \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} \dfrac{\langle \vec{a}_1, \vec{b} \rangle}{\|\vec{a}_1\|^2} \\ \dfrac{\langle \vec{a}_2, \vec{b} \rangle}{\|\vec{a}_2\|^2} \end{bmatrix}$$

Yay!! :)

Note:

$$A\vec{x}' = x_1'\vec{a}_1 + x_2'\vec{a}_2$$
$$= \frac{\langle \vec{a}_1, \vec{b} \rangle}{\|\vec{a}_1\|^2}\vec{a}_1 + \frac{\langle \vec{a}_2, \vec{b} \rangle}{\|\vec{a}_2\|^2}\vec{a}_2$$
$$= \text{proj}_{\vec{a}_1}(\vec{b}) + \text{proj}_{\vec{a}_2}(\vec{b})$$

✓

c] Compute least-squares sol'n $\vec{x}' \in \mathbb{R}^2$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$\vec{x}' = (A^T A)^{-1} A^T \vec{b}$$

$$x_1' \vec{a}_1 = \text{proj}_{\vec{a}_1} \vec{b} = \frac{\langle \vec{a}_1, \vec{b} \rangle}{\| \vec{a}_1 \|^2} \vec{a}_1 = \frac{\overbrace{1\cdot 1 + 0\cdot 2 + 0\cdot 3}^{1}}{\underbrace{1\cdot 1 + 0\cdot 0 + 0\cdot 0}_{1}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2' \vec{a}_2 = \text{proj}_{\vec{a}_2} (\vec{b}) = \frac{\langle \vec{a}_2, \vec{b} \rangle}{\| \vec{a}_2 \|^2} = \frac{\overbrace{0\cdot 1 + 1\cdot 2 + 1\cdot 3}^{5}}{\underbrace{0\cdot 0 + 1\cdot 1 + 1\cdot 1}_{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 1 \\ 5/2 \end{bmatrix}$$